平摊分析
Amortized analysis

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Implement Queue with CircularArray

- `CircularArray` supports Queue operations in $O(1)$ time.
- Recall that when the array is full:
  - Allocate a new array of **double size**.
  - Copy existing items to the new array, and insert new element.
  - Delete old array.

But now the Insert operation may take $\Theta(n)$ time.
So a sequence of n operations can take $O(n^2)$ time?

Not tight!

Amortized Analysis

- Technique for analyzing “average cost”:
  - Often used in data structure analysis
  - Idea: even when expensive operations must be performed, it is often possible to get away with performing them rarely, so that the “average” cost per operation is not so high.
  - **Note**: **Amortized analysis** is different from what is commonly referred to as average case analysis, because it does **not** make any assumption about the distribution of the data values, whereas average case analysis assumes the data are not “bad".
    - That is, amortized analysis is a **worst case** analysis, but for **a sequence of operations**, rather than for **individual operations**.

Aggregate method for Amortized Analysis

- One assumes that there is no need to distinguish between the different operations on the data structure.
  - Then we just use *aggregate method*: add up the cost of all the operations and then divide by the number of operations.
  - aggregate cost $= \dfrac{\text{maximum amount of work done by any series of } m \text{ operations}}{m}$
  - Let $N = 2^k$ for some constant k. The maximum copying cost of N insertions in `CircularArray` is $1 + 2 + 4 + \ldots + 2^{k-1} = 2^k - 1 \approx N$

Ignore cost of array alloc and free for now

  - Therefore, the aggregate cost is $O(\dfrac{N+N}{N}) = O(1)$

Amortized Analysis

- Note: **Different** operations may have **different** amortized costs.
- **Definition**: Operations has amortized cost $\hat{c}(n)$, if for every $k \in \mathbb{N}^+$, the total cost of any k operations is $\leq \sum_{i=1}^{k} \hat{c}(n_i)$.
  - n_i is the size of the data structure when applying the i^{th} operation.

Amortized Analysis

- Consider a sequence operations:
  - c_i = actual cost of the i^{th} operation; $\hat{c}_i$ = amortized cost of the i^{th} operation.
- For the amortized cost to be valid:
  - $\sum_{i=1}^{k} c_i \le \sum_{i=1}^{k} \hat{c}_i$ for any $k \in \mathbb{N}^+$.

Total cost of k operations is $\le \sum_{i=1}^{k} \hat{c}_i$

Average cost of k operations is $\le \dfrac{\sum_{i=1}^{k} \hat{c}_i}{k}$

Amortized Analysis

- Definition: Operations has amortized cost $\hat{c}(n)$, if for every $k \in \mathbb{N}^+$, the total cost of any k operations is $\leq \sum_{i=1}^{k} \hat{c}(n_i)$.
  - n_i is the size of the data structure when applying the i^{th} operation.
  - Different operations may have different amortized costs.
    - Consider CircularArray implementation of Queue.

Does *Insert* have amortized c ? ($\hat{c}(n) = 2$ if operation is **Insert**.)
Does *Remove* has amortized c ? ($\hat{c}(n) = 1$ if operation is **Remove**.)

Suppose we do not shrink array now

Ignore cost of array alloc and free for now

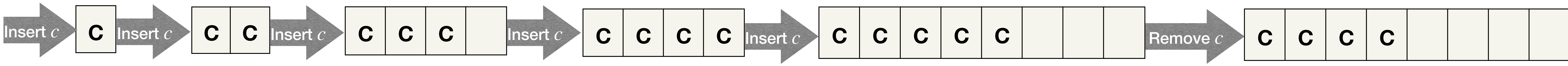
Amortized Analysis

- Definition: Operations has amortized cost $\hat{c}(n)$, if for every $k \in \mathbb{N}^+$, the total cost of any k operations is $\leq \sum_{i=1}^{k} \hat{c}(n_i)$.
  - n_i is the size of the data structure when applying the i^{th} operation.
  - Different operations may have different amortized costs.
    - Consider CircularArray implementation of Queue.

Does *Insert* have amortized cost **3**? ($\hat{c}(n) = 3$ if operation is **Insert**.)
Does *Remove* has amortized cost 1? ($\hat{c}(n) = 1$ if operation is **Remove**.)

Ignore cost of array alloc and free for now

So CircularArray operations has $O(1)$ amortized cost?
Even though some operation can cost $\Theta(n)$?



Actual total cost	1	1+(1+1)= 3	3+(2+1) = 6	6+1 = 7	7+(4+1) = 12	12 + 1 =13
Amortized total cost	3	3+3 = 6	6+3 = 9	9+3 = 12	12+3 = 15	15 + 1 = 16

Amortized Analysis Techniques

The Accounting Method

- Consider a sequence operations: c_i = actual cost of the i^{th} operation; $\hat{c}_i$ = amortized cost of the i^{th} operation. Then, the amortized cost to be valid: $\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} \hat{c}_i$ for any $k \in \mathbb{N}^+$.
- Imagine you have a bank account B with initial balance 0.
- For the i^{th} op., you spend $\hat{c}_i$ money:
  - Recall that the actual cost of the i^{th} op. is c_i
  - If $\hat{c}_i \geq c_i$, pay c_i for the op., and **deposit** $\hat{c}_i - c_i$ into B.
  - If $\hat{c}_i < c_i$, pay c_i for the op., and **withdraw** $c_i - \hat{c}_i$ into B.
- Amortized analysis valid if $B = \Sigma_{i=1}^{k}(\hat{c}_i - c_i)$ always ≥ 0

Example: CircularArray based Queue

- $\hat{c}_i = 3$ if the i^{th} operation is **Insert,** $\hat{c}_i = 1$ if the i^{th} operation is **Remove.**
- **Goal:** Prove $\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} \hat{c}_i$ for any $k \in \mathbb{N}^+$ operations.
- **Strategy**: account always non-negative via induction on k.
  - **[Basis]** Prior to 1^{st} op., account balance is 0.
  - **[Hypothesis]** Prior to i^{th} op., account balance is always non-negative.

Example: CircularArray based Queue

- **[Inductive Step]** Consider the i^{th} op.
  - If it's **Remove**, then we make no change to account value.

    $\hat{c}_i - c_i = 1 - 1 = 0$

  - If it's **Insert** without expansion, we add 2 to account value.

    $\hat{c}_i - c_i = 3 - 1 = 2$

  - If it's **Insert** with expansion. Assume *expand* from n to $2n$, then we need to withdraw $n - 1$ value

    $c_i - \hat{c}_i = n + 1 - 2 = n - 1$

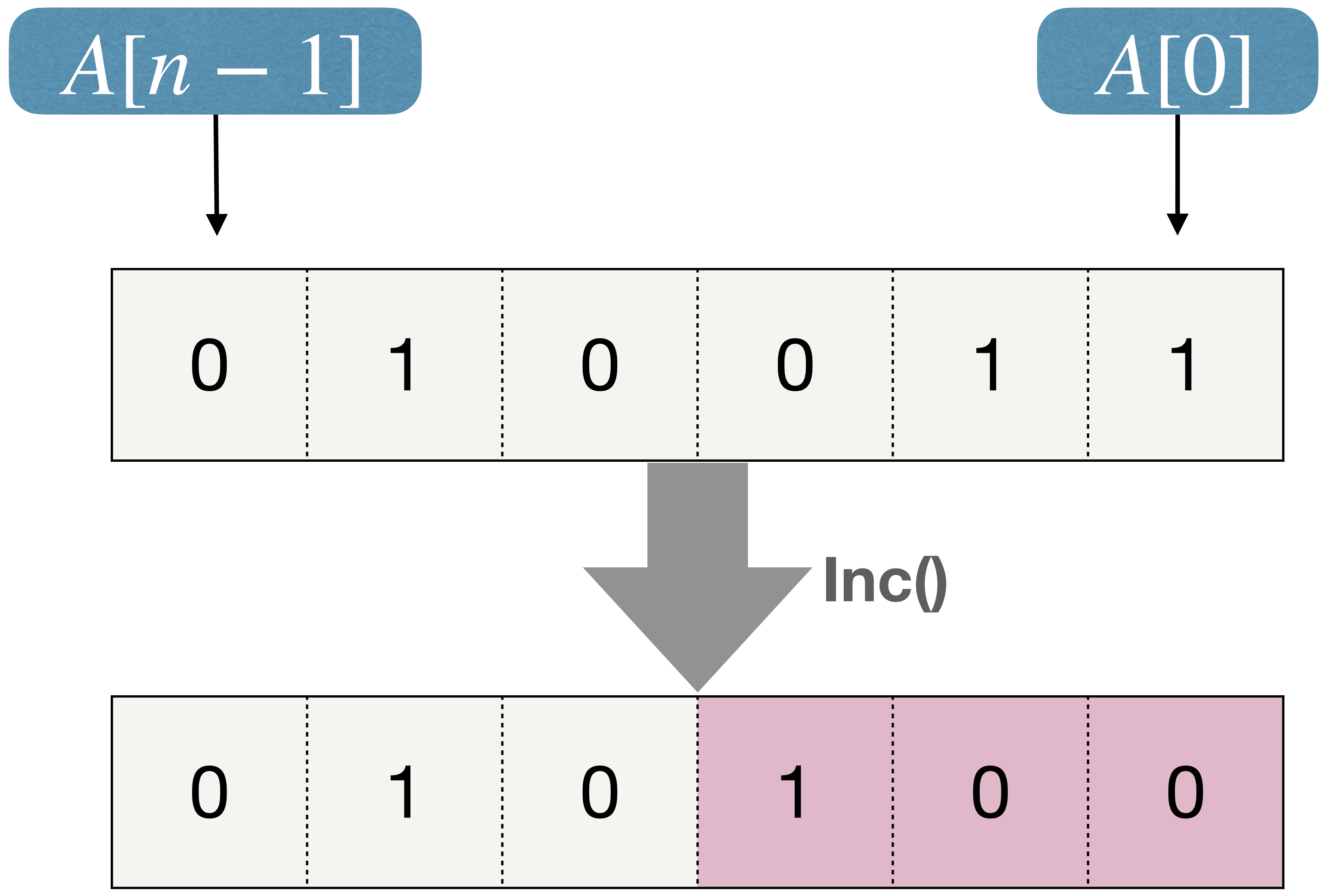
    - Last *expand* must be from $n/2$ to n.
    - Since last *expand*, each **Insert** adds 2, each **Remove** makes no change.
    - Since last *expand*, there are at least $n/2$ **Insert** op.
    - Immediately after last *expand*, account balance is non-negative (**Hypothesis**).
    - Thus prior to i^{th} op., account balance $\geq n$. This is enough!

Example: Binary Counter

- Use length n binary array A to represent a number.
- The number is 0 initially, and ***Inc*** op. adds 1 to this number.
- Cost of In ***Inc***: number of bits it flipped.
- Average cost of k ***Inc*** operations?
  - Easy answer: $O(n)$
  - More careful analysis… (Amortized analysis…)

Inc(A):

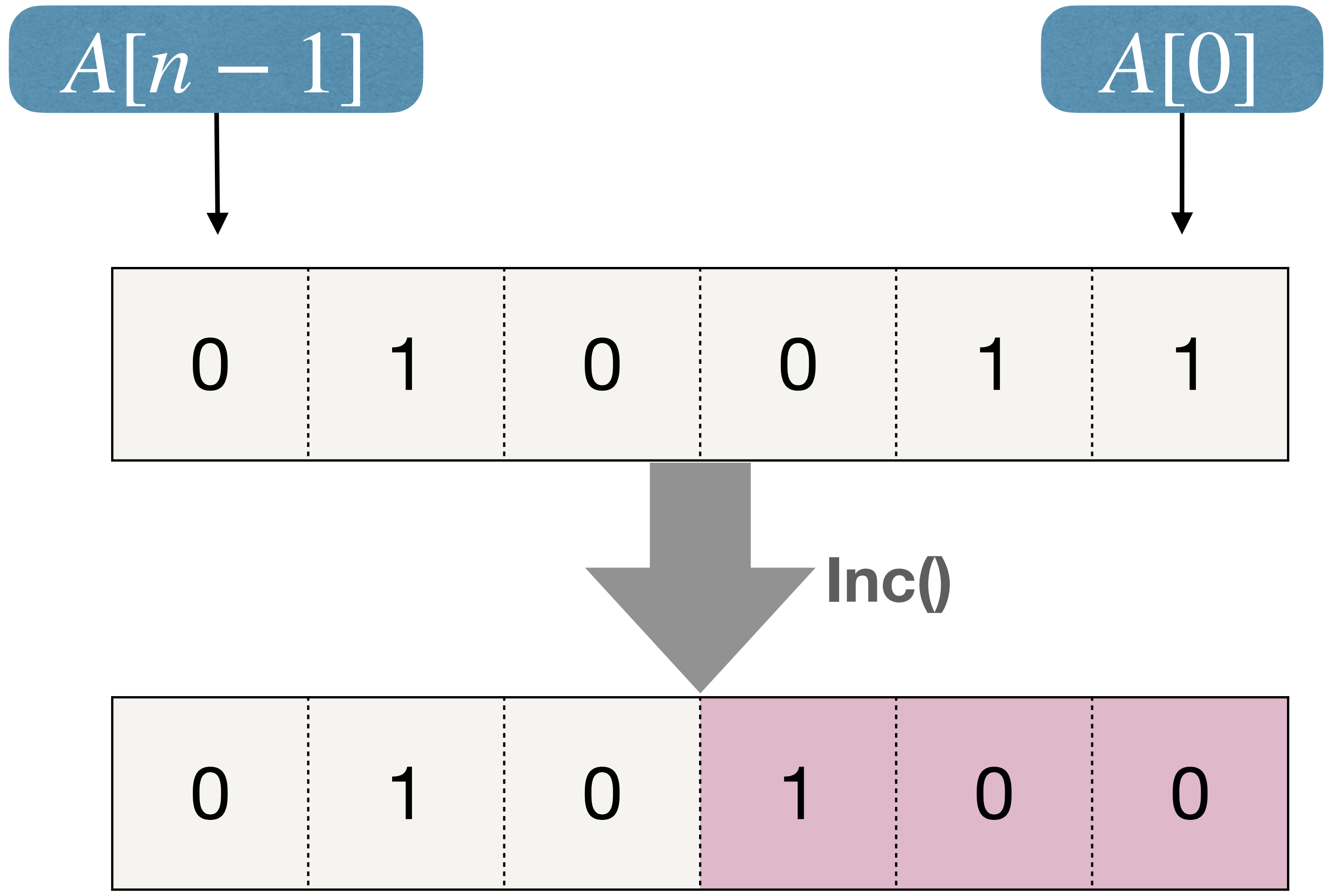
$i := 0$
while $i < n$ **and** $A[i] = 1$
    $A[i] := 0$
    $i := i + 1$
if $i < n$
    $A[i] := 1$



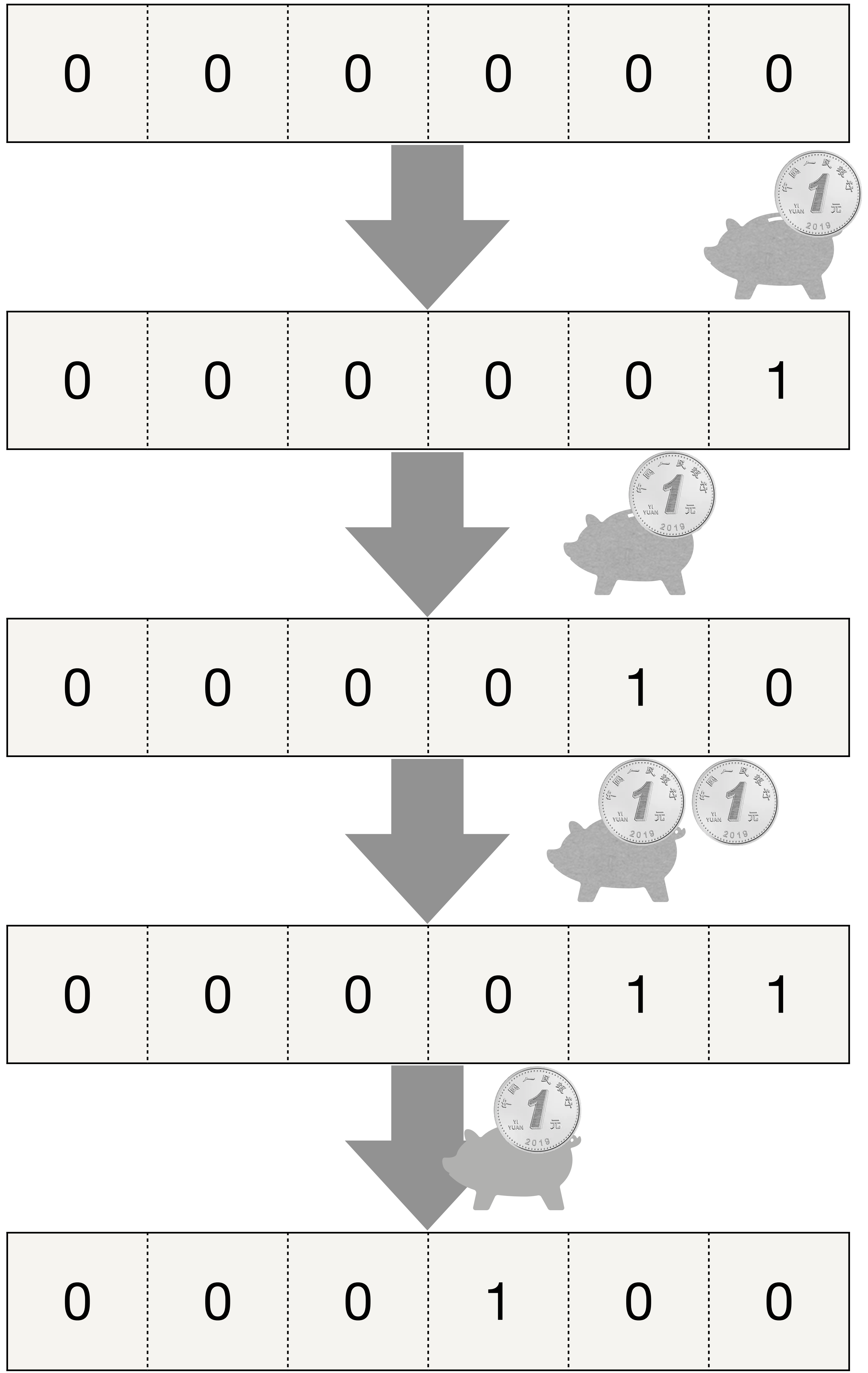
Example: Binary Counter

- The number is 0 initially, and ***Inc*** op. adds 1 to this number.
- Cost of In ***Inc***: number of bits it flipped.
- In each ***Inc***: $0 \to 1$: at most 1 bit, while $1 \to 0$: many bits!
- **But a bit has to be set to 1 before it resets to 0!**
- If we **deposit** 1 whenever we $0 \to 1$, later $1 \to 0$ are **"free"!**
- Each ***Inc*** does $0 \to 1$ at most once, so amortized cost is: $2 = O(1)$

Inc(A):
$i := 0$
while $i < n$ **and** $A[i] = 1$
    $A[i] := 0$
    $i := i + 1$
if $i < n$
    $A[i] := 1$



Example: Binary Counter



Actual total cost	Amortized total cost
1	$\times 2$
$1+2=3$	$\times 4$
$3+1=4$	$\times 6$
$4+3=7$	$\times 8$

The Potential Method

- Consider a sequence operations: c_i = actual cost of the i^{th} operation; $\hat{c}_i$ = amortized cost of the i^{th} operation. Then, the amortized cost to be valid: $\sum_{i=1}^{k} c_i \leq \sum_{i=1}^{k} \hat{c}_i$ for any $k \in \mathbb{N}^+$.

Now let us consider the amortized cost in a higher level than the specific value in one operation (accounting)!

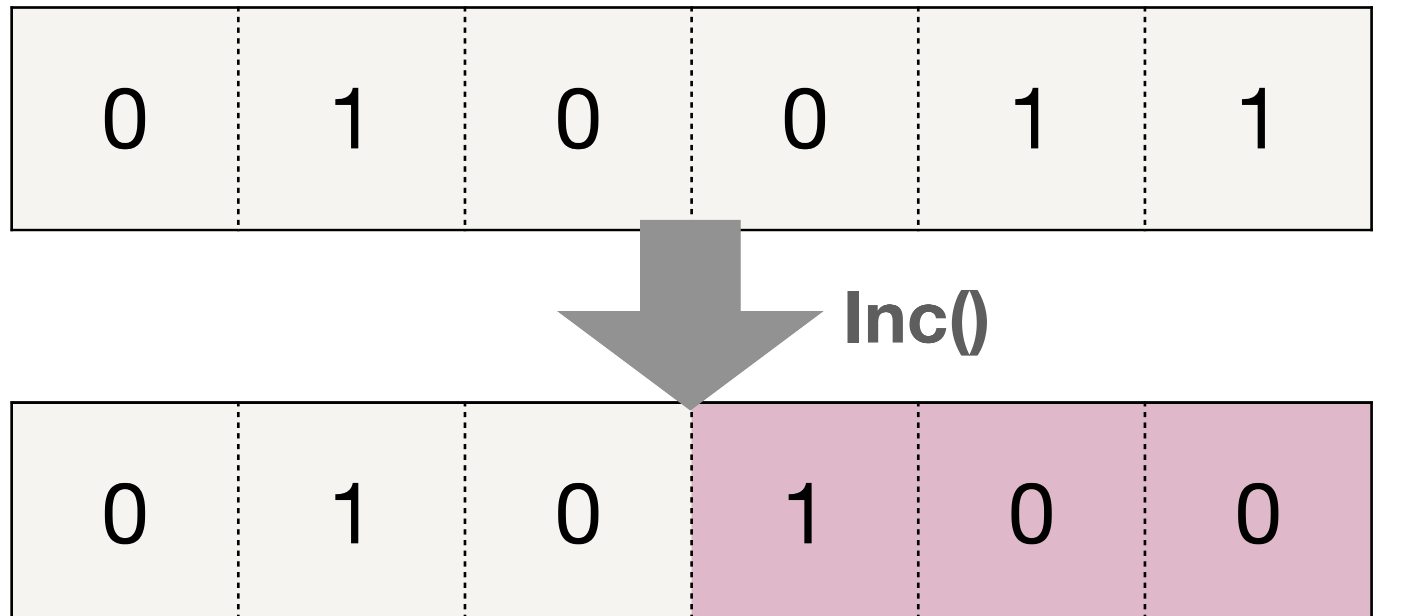
- Design a **potential function** Φ that maps data structure status to real values
  - $\Phi(D_0)$: initial potential of the data structure, usually set to 0.
  - $\Phi(D_i)$: potential of the data structure after i^{th} operation.
- Define $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$
- For amortized cost to be valid, need $\Phi(D_k) \geq \Phi(D_0)$ for all k.

The Potential Method

- “Potential” is like the balance in account in “Counting Method”.
  - Potential slowly accumulates during “cheap” operations (deposit).
  - Potential drops a lot after an “expensive” operation (withdraw).
- But the Potential Method could be more powerful in general…

Example: Binary Counter

- How to define $\Phi(D_i)$ for Binary Counter? (Recall potential is like “balance”.)
- $\Phi(D_i)$ = number of 1s in the array after the i^{th} ***Inc*** operation.
- Clearly “$\Phi(D_k) \geq \Phi(D_0)$ for all k.” is satisfied, how large is $\hat{c}_i$?
  - c_i = (number of bits $0 \rightarrow 1$) + (number of bits $1 \rightarrow 0$)
  - $\Phi(D_i) - \Phi(D_{i-1})$ = (number of bits $0 \rightarrow 1$) - (number of bits $1 \rightarrow 0$)
- $\hat{c}_i = 2 \cdot$ (number of bits $0 \rightarrow 1$) ≤ 2



Back to CircularArray based Queue

- Now suppose we need to shrink array for space consideration
  - **Solution(1):** Reduce array size to half when array only half loaded after Remove. (Allocate new array of half size, copy items to new array, and delete old array.)
  - **Solution(2):** Reduce array size to half when array only $1/4$ loaded after Remove. (Allocate new array of half size, copy items to new array, and delete old array.)
- Quiz: which one is better with respect to amortized cost?

Further reading

- [CLRS] Ch.17