智能软件与工程学院
School of Intelligent Software and Engineering

图及其遍历
Graphs and Graph Traversal

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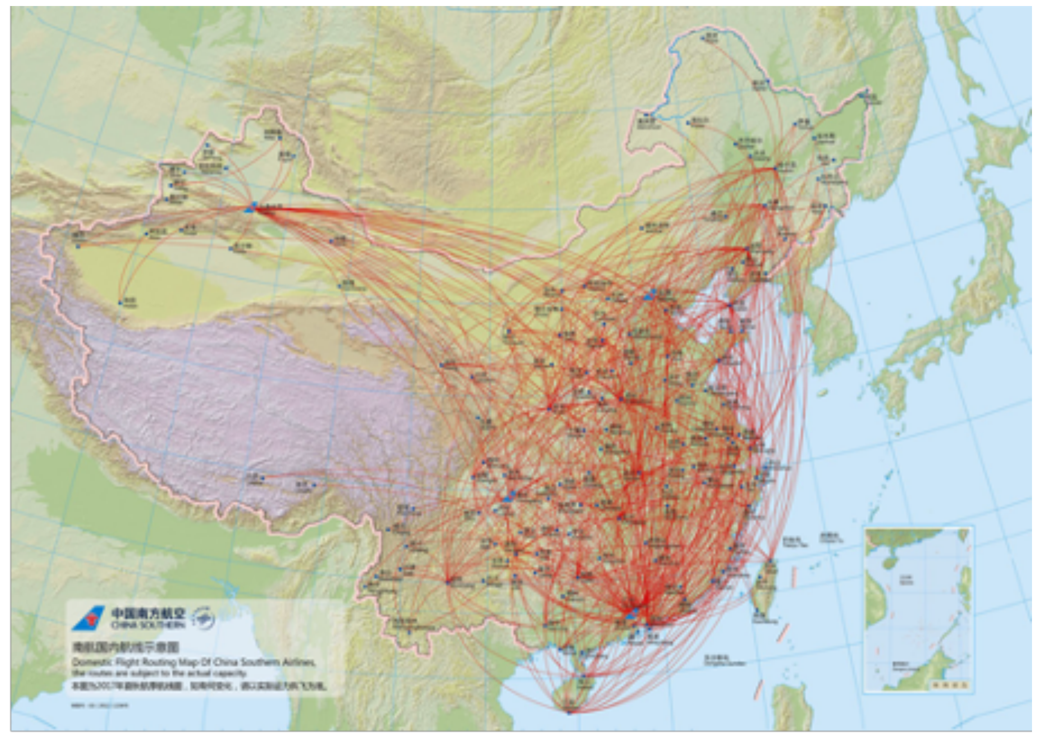
2023 Fall

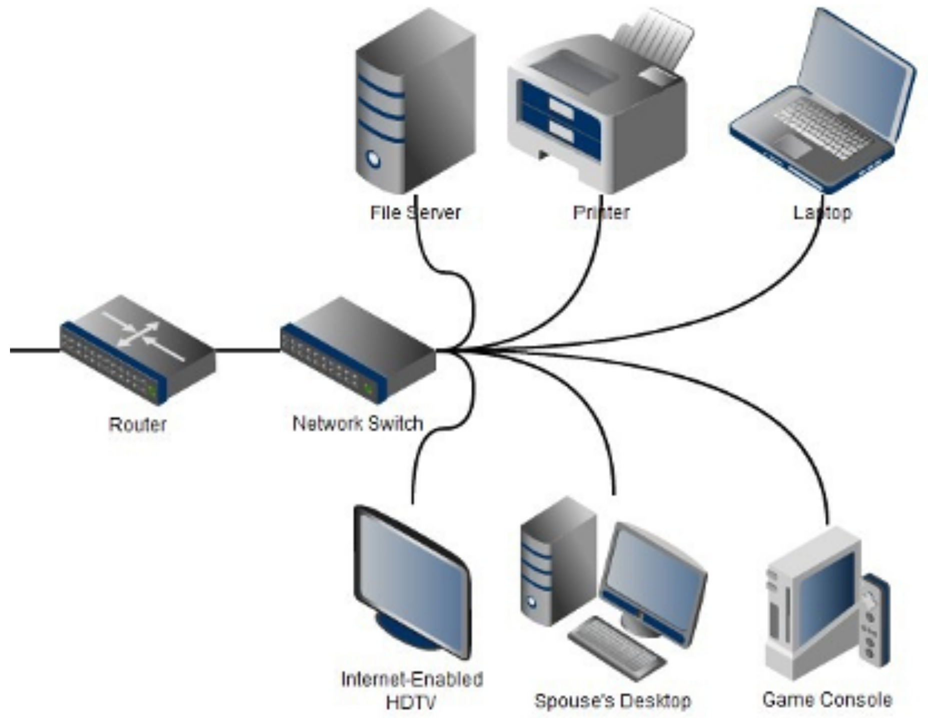
The slides are mainly adapted from the original ones shared by Chaodong Zheng and Kevin Wayne. Thanks for their supports!

Graphs are Everywhere!

- **Transportation Networks.**
  - Nodes: Airports; Edges: Nonstop flights.



- **Communication Networks.**
  - Nodes: Computers; Edges: Physical links.



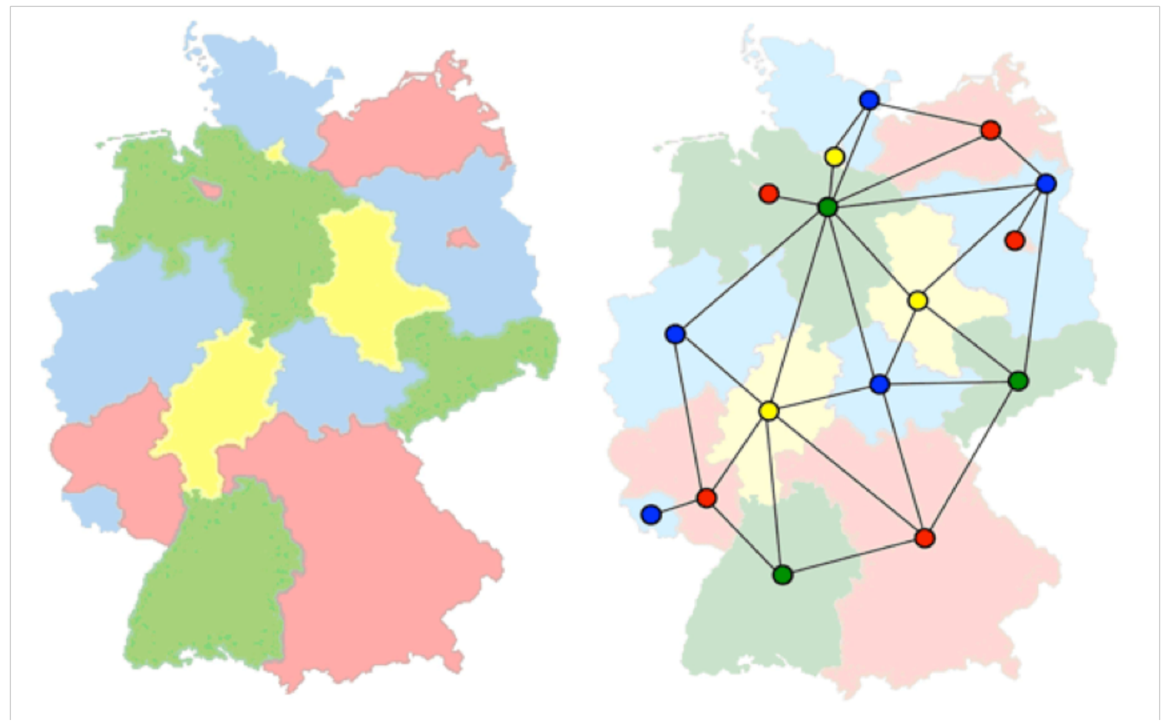
- **Social Networks.**
  - Nodes: People; Edges: Friendship.

- …

Followed by many graph problems

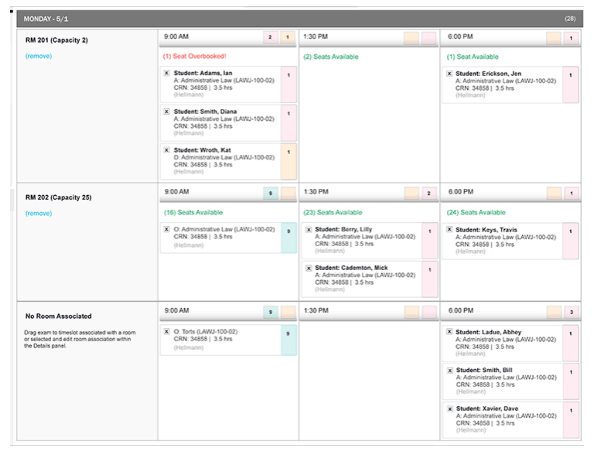
Coloring Maps

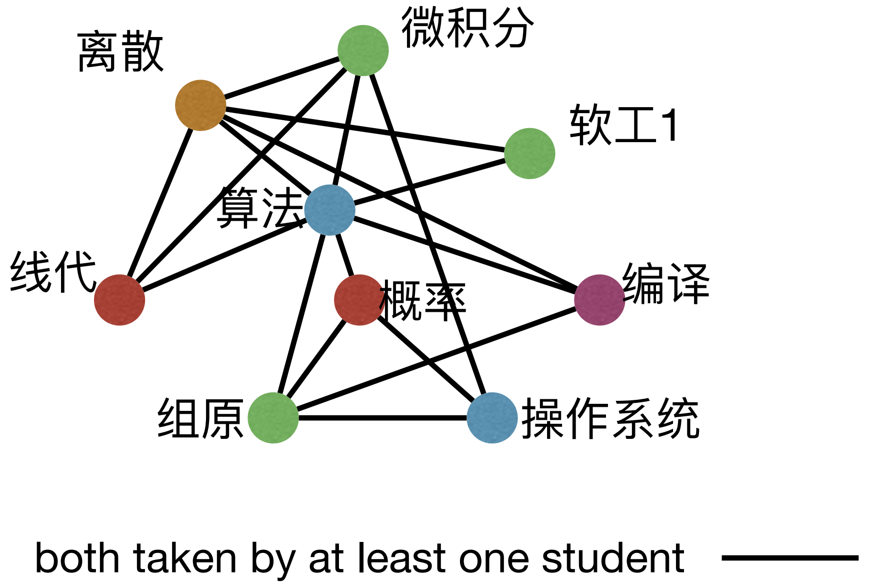
- Nodes: Countries; Edges: Neighboring countries.
- Question of Interest: Chromatic number?



Scheduling Exams

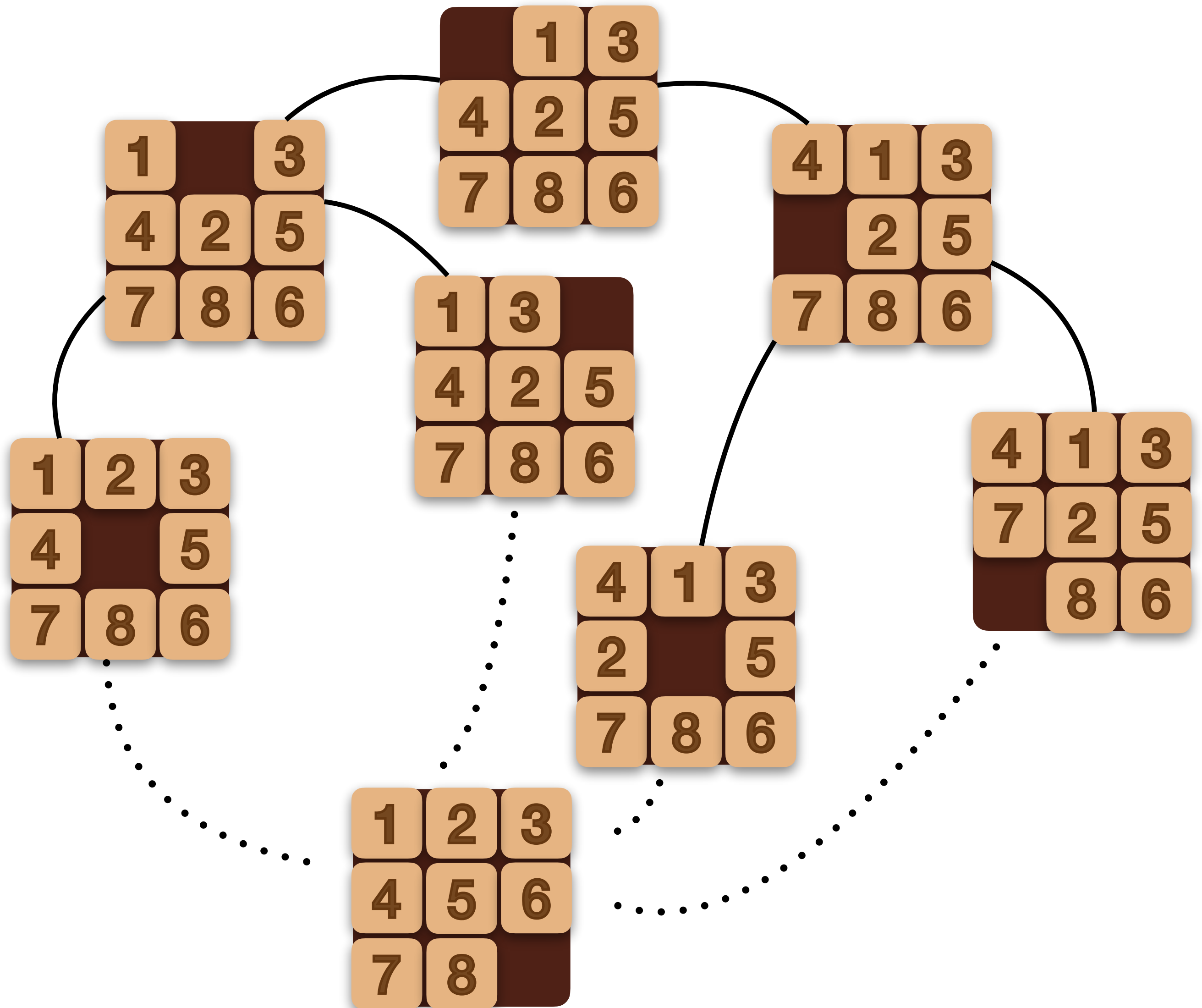
- Nodes: Exams; Edges: Conflicts.
- Question of Interest: Chromatic number?





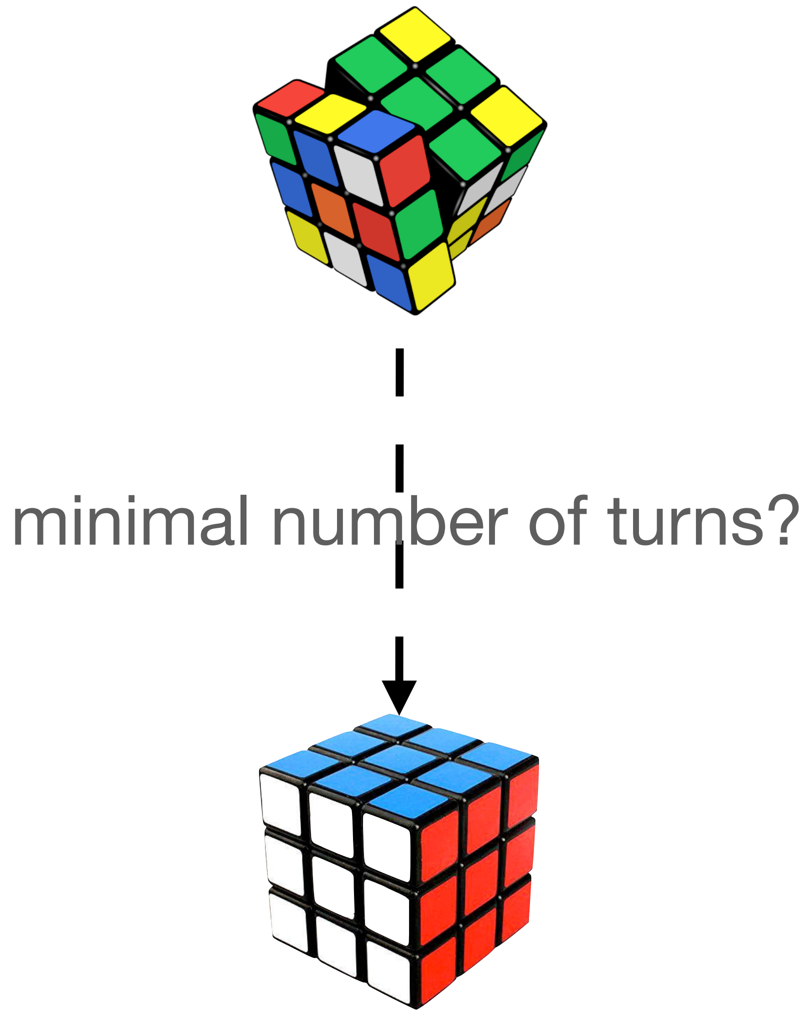
Solving Sliding Puzzle

- Nodes: States; Edges: Legit moves
- Question of Interest: Shortest path?



Solving Rubik's Cube

- Nodes: States; Edges: Legit moves
- Question of Interest: God's Number?



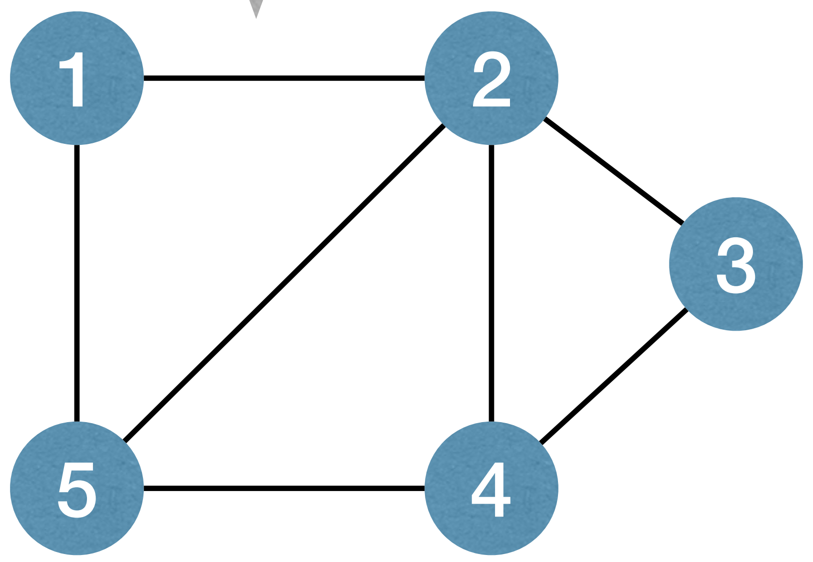
Representing graphs in computers

- **Adjacency Matrix**

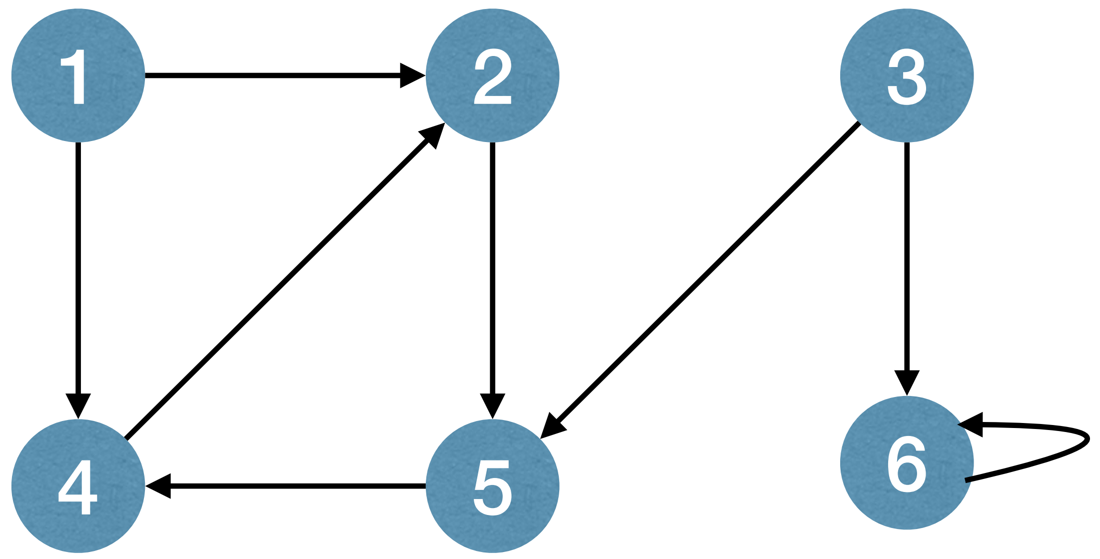
  - Consider a graph $G = (V, E)$, where $|V| = n$ and $|E| = m$
  - The Adjacency Matrix of G is an $n \times n$ matrix $A = (a_{ij})$ where
    - $a_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$
  - The matrix will be **symmetry** if G is undirected.
  - The matrix will always cost $\Theta(n^2)$ memory, regardless of m.

Quick Question: What does A^2 mean, if anything?

Simple Graph without self loop



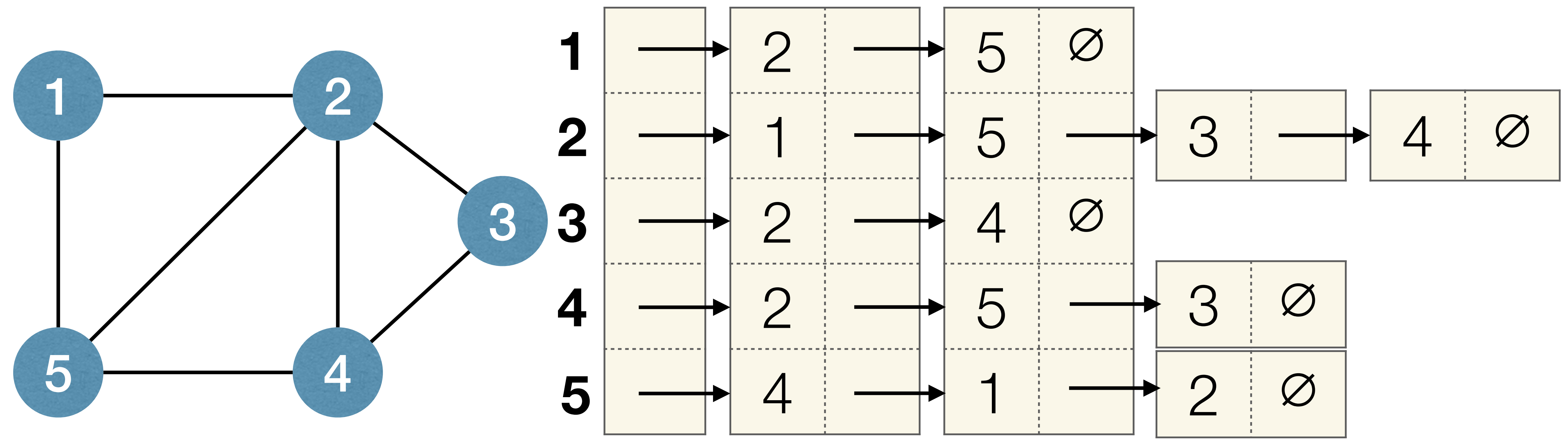
	1	2	3	4	5
1	0	1	0	0	1
2	1	0	1	1	1
3	0	1	0	1	0
4	0	1	1	0	1
5	1	1	0	1	0

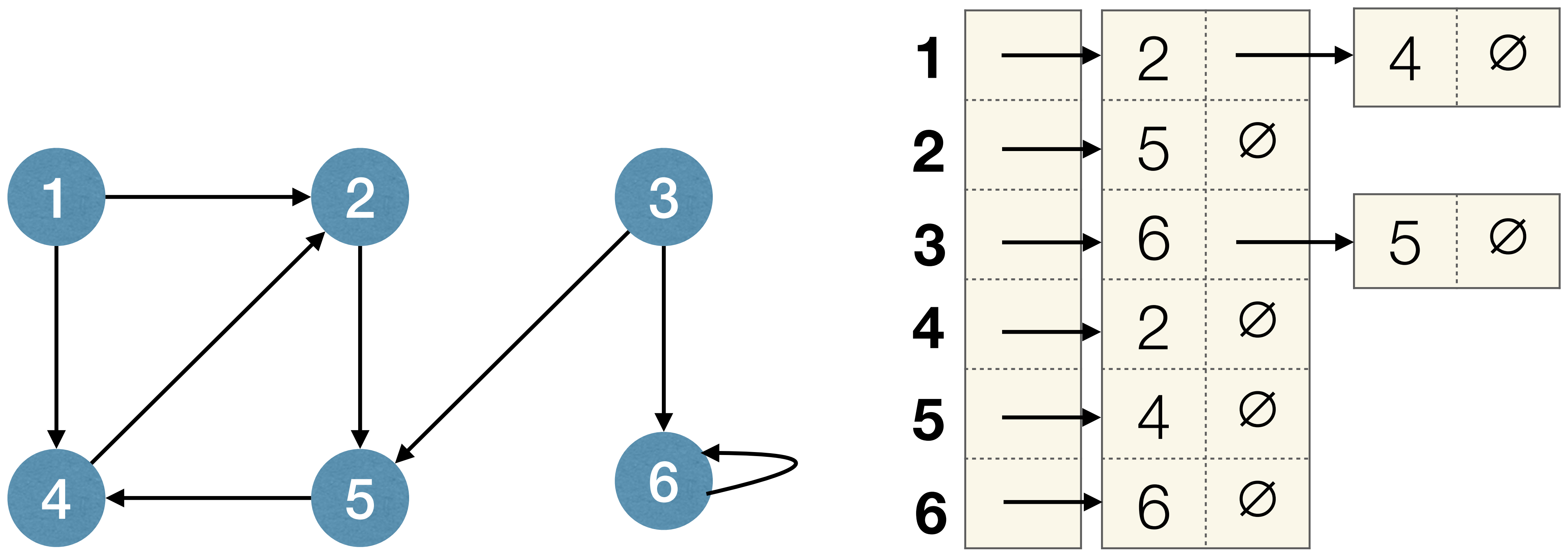


	1	2	3	4	5	6
1	0	1	0	1	0	0
2	0	0	0	0	1	0
3	0	0	0	0	1	1
4	0	1	0	0	0	0
5	0	0	0	1	0	0
6	0	0	0	0	0	1

Representing graphs in computers

- **Adjacency List**
  - Consider a graph $G = (V, E)$, where $|V| = n$ and $|E| = m$
  - The Adjacency List of G is a collection of n lists:
    - One for each vertex $u \in V$
    - In the list for u, vertex v exists iff edge $(u, v) \in E$
  - Each edge appears twice if G is undirected.
  - The space cost is $\Theta(n + m)$





Adjacency Matrix and Adjacency List

Trade-offs

Adjacency Matrix

- **Fast Query**: Are u and v neighbors?
- **Slow Query**: Find me any neighbor of u.
- **Slow Query**: Enumerate all neighbors of u.

Adjacency List

- **Fast Query**: Find me any neighbor of u.
- **Fast Query**: Enumerate all neighbors of u.
- **Slow Query**: Are u and v neighbors?

Queries: What types of queries are needed and/or frequent?
Space usage: Is the graph dense or sparse?

Important question to ask

Searching in a Graph (or, Graph Traversal)

- **Goal**: Start at source node s and find some node t.
- **Or**: Visit all nodes reachable from s.
- Two Basic Strategies:
  - **Breadth-First Search (BFS)**
  - **Depth-First Search (DFS)**
- Many applications, beside searching and traversal!
- Usually use adjacency list when discussing BFS/DFS. (At least in this course…)

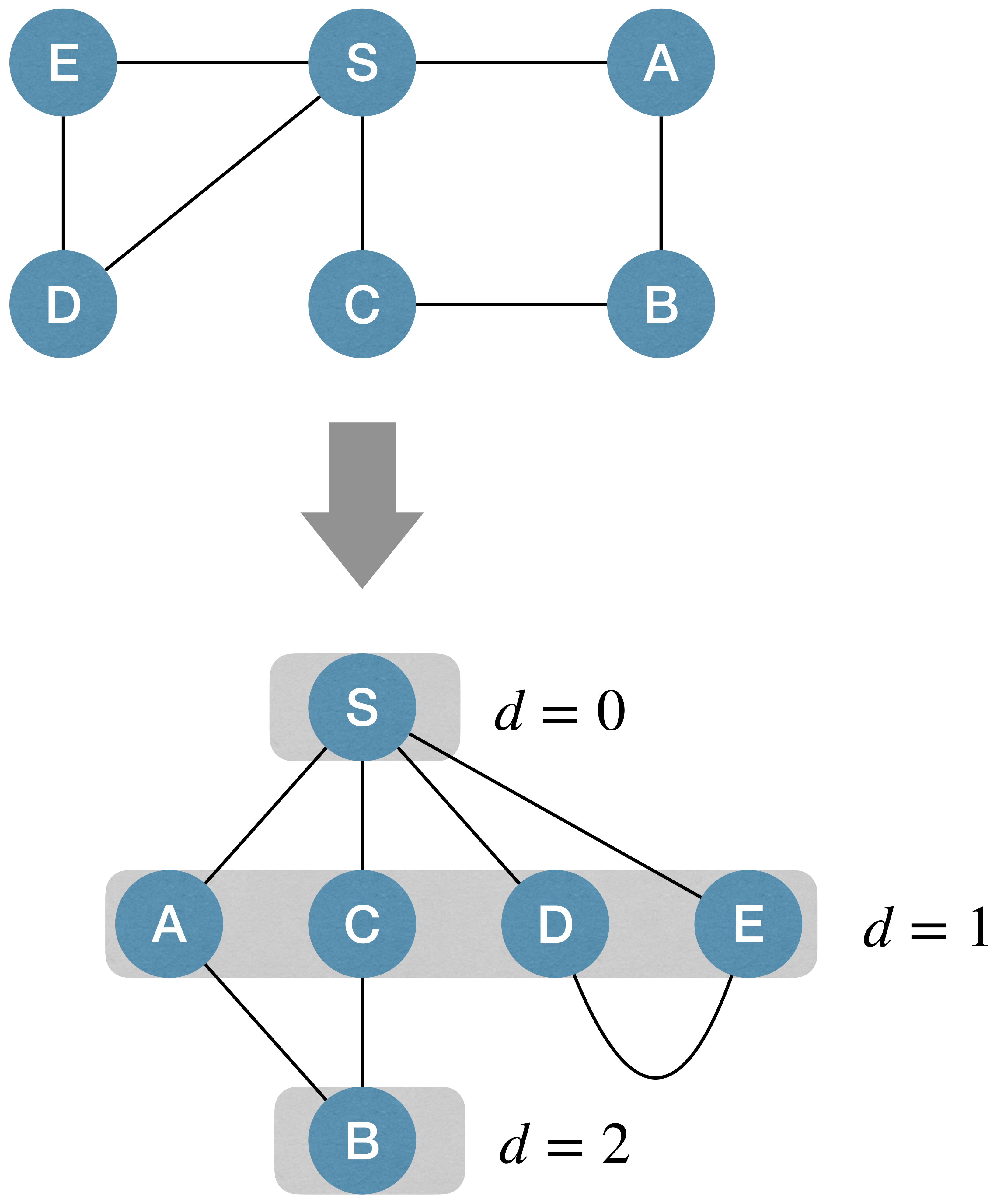
Breadth-First Search

Breadth-First Search (BFS)

- Basic Idea of BFS:
  - Start at the source node s;
  - Visit other nodes (reachable from s) "*layer by layer*".
- More precise description:
  - Start at the source node s;
  - Visit nodes at *distance* 1 from s;
  - Visit nodes at *distance* 2 from s;
  - …

Visit all distance d nodes, before visit any distance $d+1$ node.

These nodes are neighbors of distance 1 nodes!



BFS Implementation

- How to implement BFS? (Hint: recall traversal-by-layer in trees)
  - Use a FIFO Queue!
- Nodes have 3 status:
  - Undiscovered (WHITE): Not in queue yet.
  - Discovered but not visited (GRAY): In queue but not processed.
  - Visited (BLACK): Ejected from queue and processed.

BFSSkeleton(G, s):

for **each** u **in** V
    $u.dist := INF,\ u.discovered := False$
$s.dist := 0,\ s.discovered := True$
$Q.enque(s)$
while $!Q.empty()$
    $u := Q.dequeue()$
    for **each** edge (u, v) **in** E
        if $!v.discovered$
            $v.dist := u.dist + 1$
            $v.discovered := True$
            $Q.enque(v)$

We can “store” a shortest path, instead of only “computing” the length of the path —> by additionally recording the node’s parent info.

BFS Implementation

```
BFS(G, s):
for each u in V
    u.c := WHITE, u.d := INF, u.p := NIL
s.c := GRAY , s.d := 0, s.p := NIL
Q.enque(s)
while !Q.empty()
    u := Q.dequeue()
    u.c := BLACK
    for each edge (u, v) in E
        if v.c = WHITE
            v.c := GRAY
            v.d := u.d + 1
            v.p := u
            Q.enque(v)
```

Sample Execution

Performance of BFS

- Runtime of BFS? (Assuming G is connected.)
- "While" loop $\Theta(n)$ times.
  - Each node in Q at most once.
- "For" loop $\Theta(m)$ times.
  - Each edge visited at most once or twice.
- Runtime of BFS is $\Theta(n + m)$.

BFS(G, s):

```
for each u in V
    u.c := WHITE, u.d := INF, u.p := NIL
s.c := GRAY, s.d := 0, s.p := NIL
Q.enque(s)
while !Q.empty()
    u := Q.dequeue()
    u.c := BLACK
    for each edge (u, v) in E
        if v.c = WHITE
            v.c := GRAY
            v.d := u.d + 1
            v.p := u
            Q.enque(v)
```

What if we use adjacency matrix instead of adjacency list?

Correctness and Properties of BFS

Theorem BFS visits a node iff it is reachable from s.

Proof:

- [*only if*] If a node is not reachable from s, then BFS does not visit it, since BFS only moves along edges.
- [*if*] If a node is reachable from s, then BFS visits it.
  - **Claim:** For all $k \geq 0$, all nodes within k hops of s are visited.
    - [*Basis*]: Clearly s is visited.
    - [*Hypothesis*]: All nodes within $k-1$ hops of s are visited.

Correctness and Properties of BFS

Theorem BFS visits a node iff it is reachable from s.

- [*Inductive Step*]: Consider a node v that is k hops away from s. Let u be v's neighbor on (one of) v's shortest path back to s

  By induction hypothesis, u gets visited.

  When BFS visits u, node v is already ***GRAY*** or ***BLACK***, or will be put in Q.

  Either way, v eventually gets visited.

Will this really happen?!

Correctness and Properties of BFS

Theorem BFS correctly computes $u.dist$, for every node u that is reachable from s

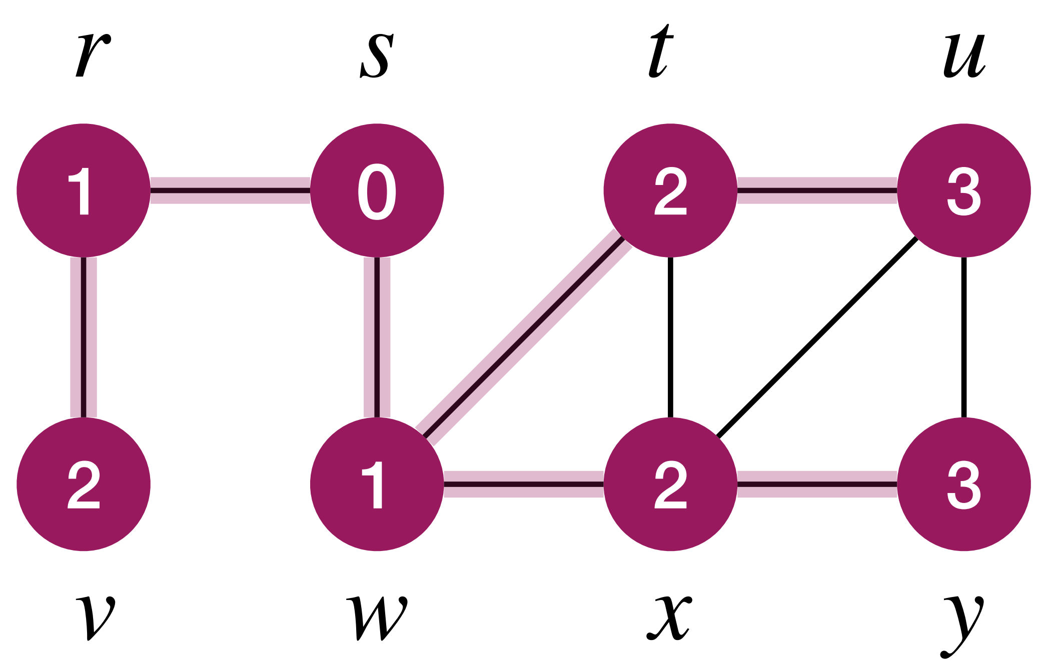
- [*Proof Idea*] Use induction to show:
  - For all $d \geq 0$, there is a moment at which:
    - (**a**) every node u with $dist(s, u) \leq d$ correctly computes $u.dist$;
    - (**b**) every other node v has $v.dist = \infty$;
    - (**c**) Q contains exactly the nodes d hops away from s.

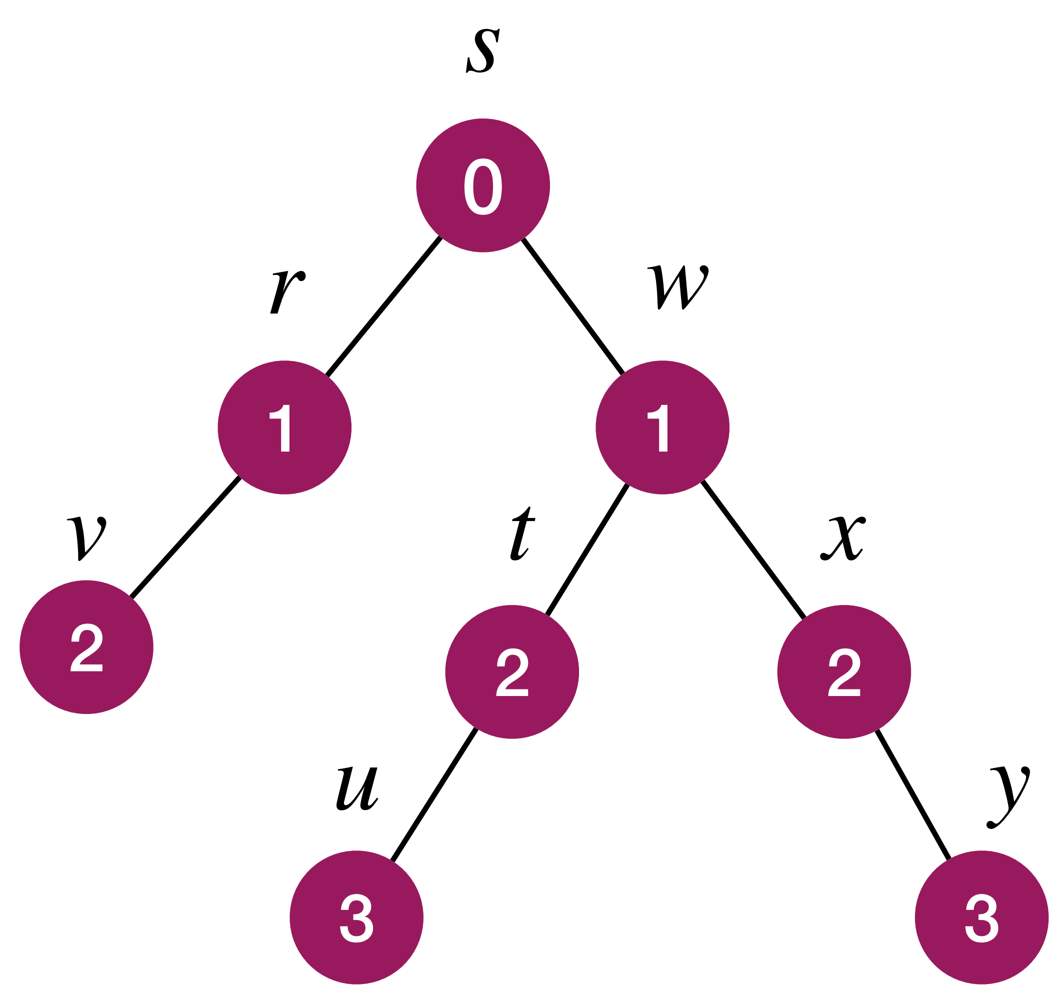
Correctness and Properties of BFS

Theorem BFS correctly computes $u.dist$, for every node u that is reachable from s

Corollary For any $u \neq s$ that is reachable from s, one of the shortest path from s to u is a shortest path from s to $u.p$ followed by the edge $(u.p, u)$

- $G_p = (V_p, E_p)$ is a **breadth-first tree**, which can print on a shortest path from any node v to the source node s. Here:
  - $V_p = \{u \in V : u.p \neq \text{NIL}\} \cup \{s\}$,
  - $E_p = \left\{(u.p, u) : u \in V_p - \{s\}\right\}$.





One last note on BFS

- What if the graph is not connected?
  - Easy, do a BFS for each connected component!

Runtime of this procedure?

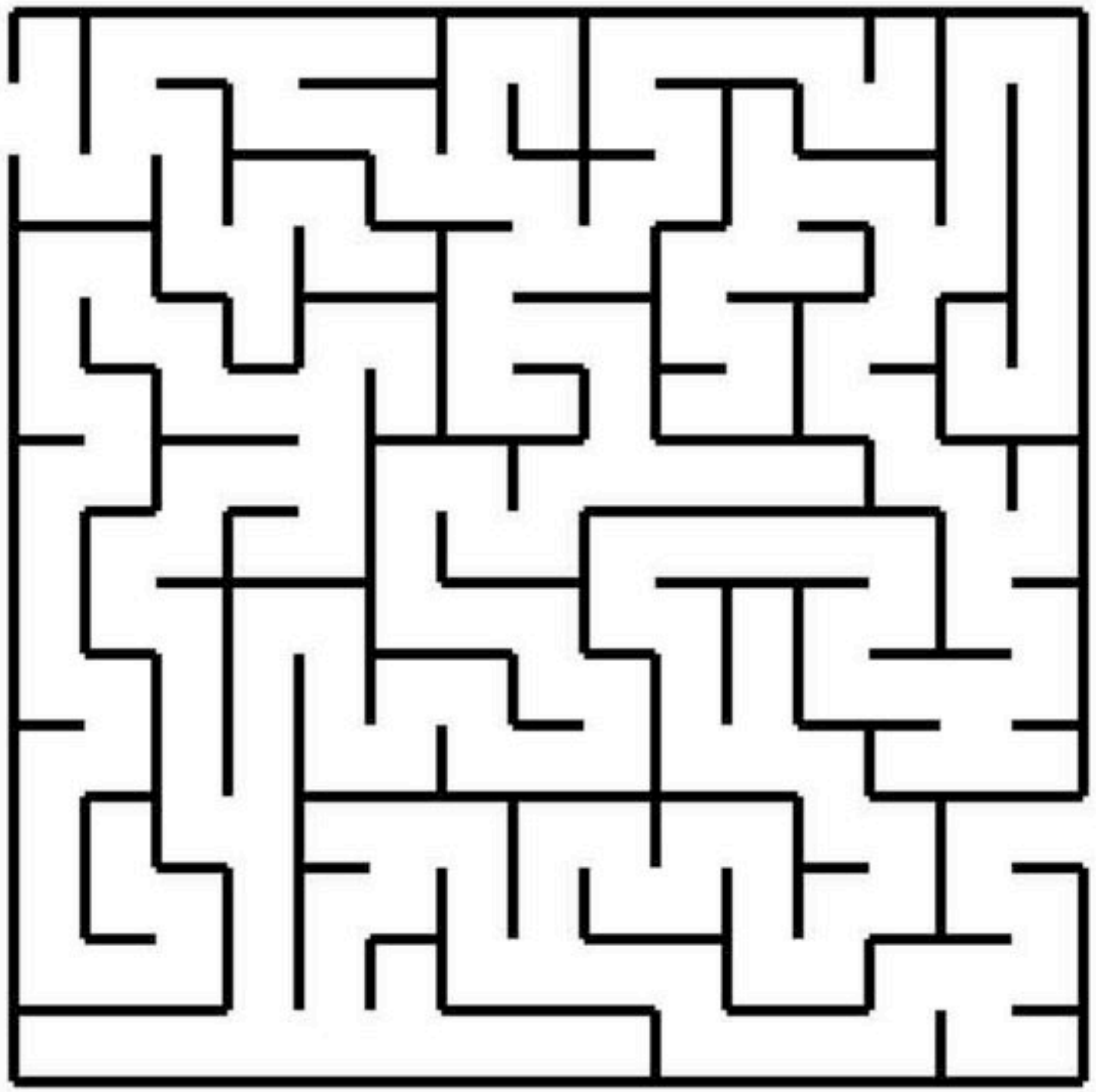
BFS(G):

for each u in V
    $u.c := WHITE, u.d := INF, u.p := NIL$
for each u in V
    if $u.c = WHITE$
        $u.c := GRAY, u.d := 0, u.p := NIL$
        $Q.enque(u)$
        while $!Q.empty()$
            $v := Q.dequeue()$
            $v.c := BLACK$
            for each edge (v, w) in E
                if $w.c = WHITE$
                    $w.c := GRAY$
                    $w.d := v.d + 1$
                    $w.p := w$
                    $Q.enque(w)$

Depth-First Search

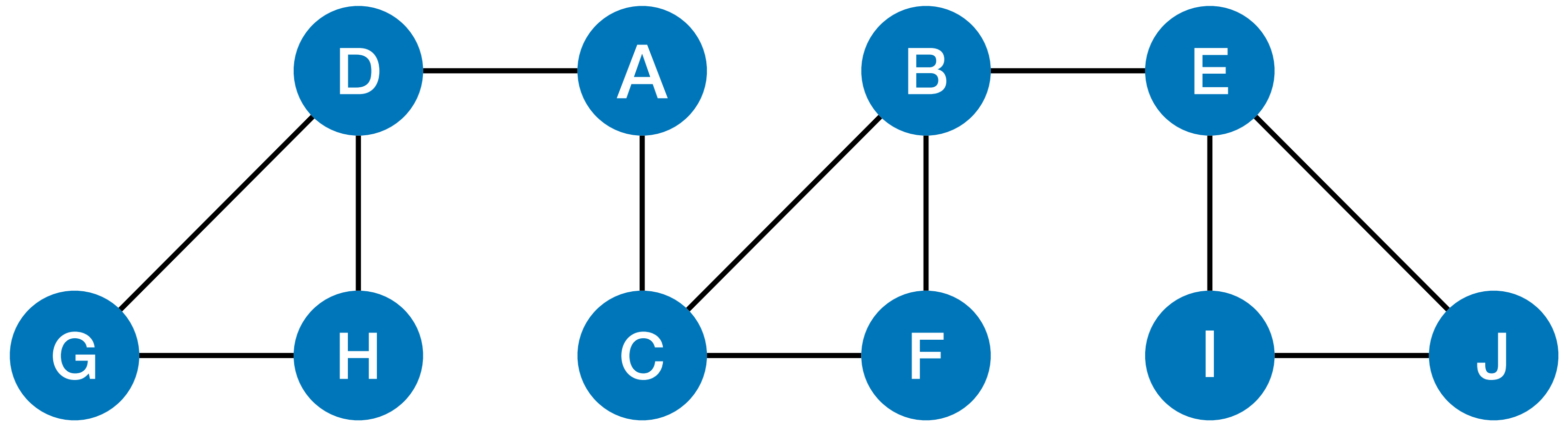
Depth-First Search (DFS)

- Much like exploring a maze:
  - Use **a ball of string** and **a piece of chalk**.
  - Follow path (unwind string and mark at intersections), until stuck (reach dead-end or already-visited place).
  - Backtrack (rewind string), until find unexplored neighbor (intersection with unexplored direction).
  - Repeat above two steps.



Depth-First Search (DFS)

- How to do this for a graph, in computer?
  - **Chalk**: boolean variables.
  - **String**: a stack.



DFSSkeleton(G, s):

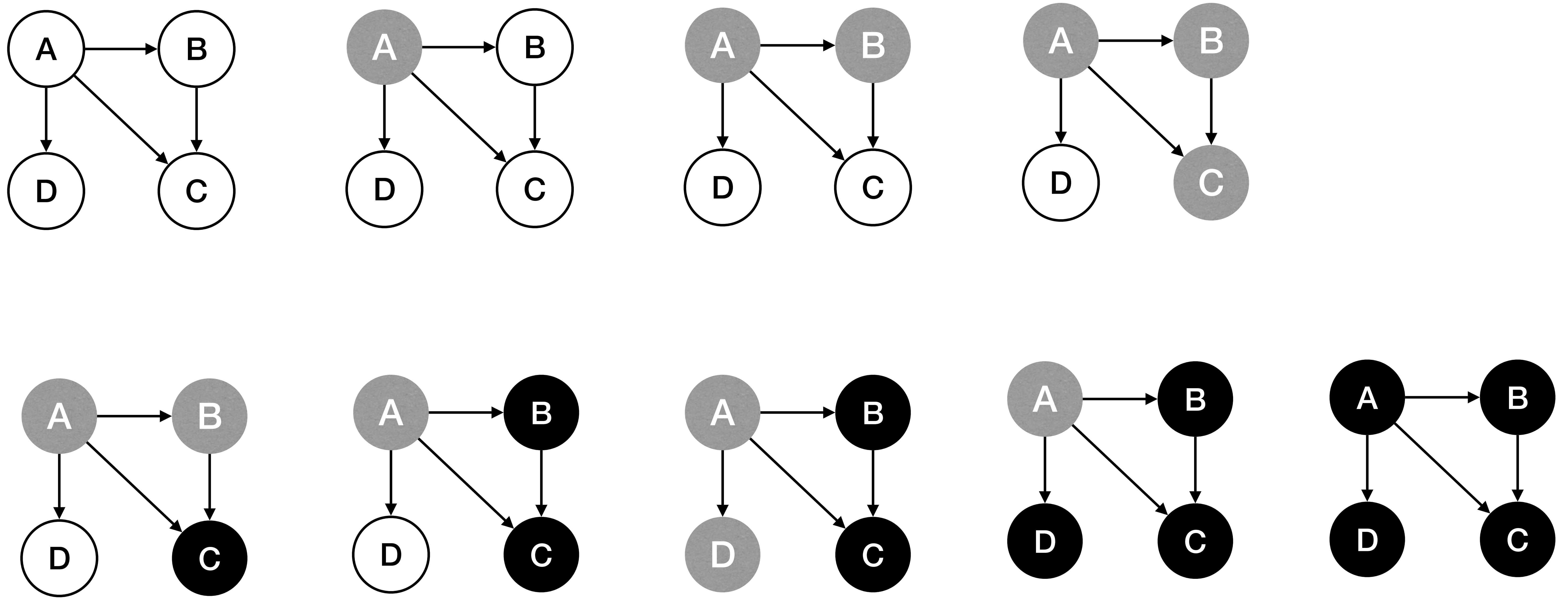
```
s.visited := True
for each edge (s, v) in E
    if !v.visited
        DFSSkelecton(G, v)
```

DFSIterSkeleton(G, s):

```
Stack Q
Q.push(s)
while !Q.empty()
    u := Q.pop()
    if !u.visited
        for each edge (u, v) in E
            Q.push(v)
```

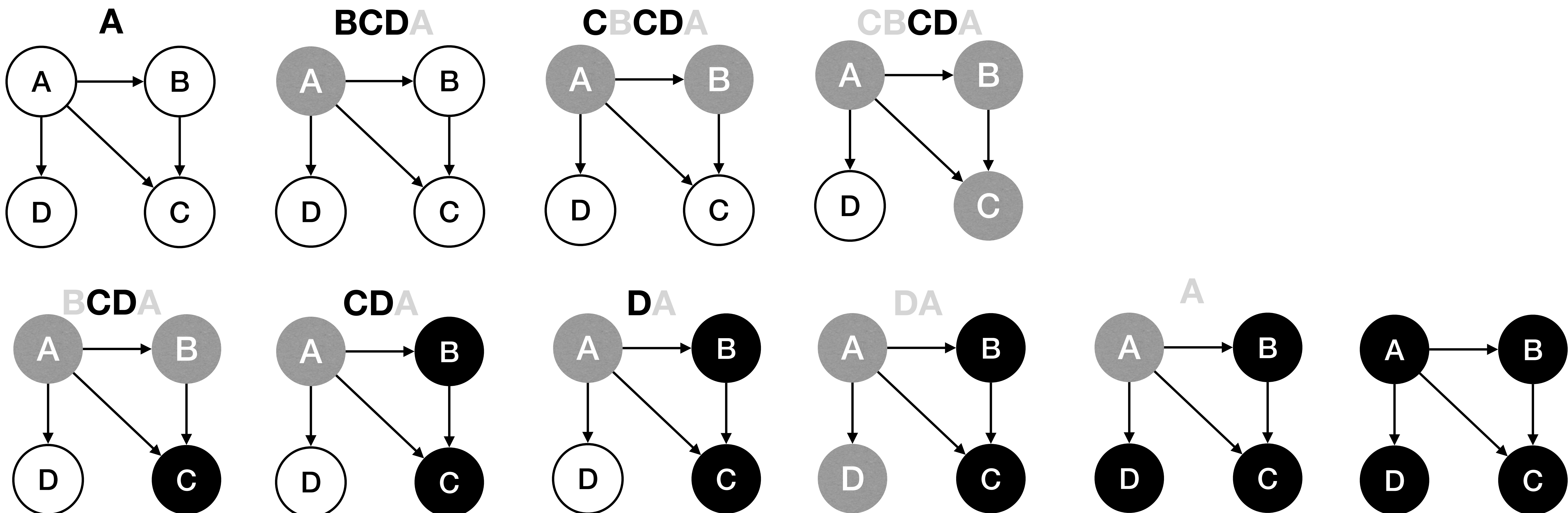
DFSSkeleton(G, s):

```
s.visited := True
for each edge (s, v) in E
    if !v.visited
        DFSSkelecton(G, v)
```



DFSIterSkeleton(G, s):

```
Stack Q
Q.push(s)
while !Q.empty()
    u := Q.pop()
    if !u.visited
        for each edge (u, v) in E
            Q.push(v)
```



Depth-First Search (DFS)

- What if the graph is not (strongly) connected?
  - Do DFS from multiple sources.

```
DFSAll(G):
for each node u in V
    v.visited := False
for each node u in V
    if !u.visited
        DFSSkelecton(G, u)
DFSSkeleton(G, s):
s.visited := True
for each edge (s, v) in E
    if !v.visited
        DFSSkelecton(G, v)
```

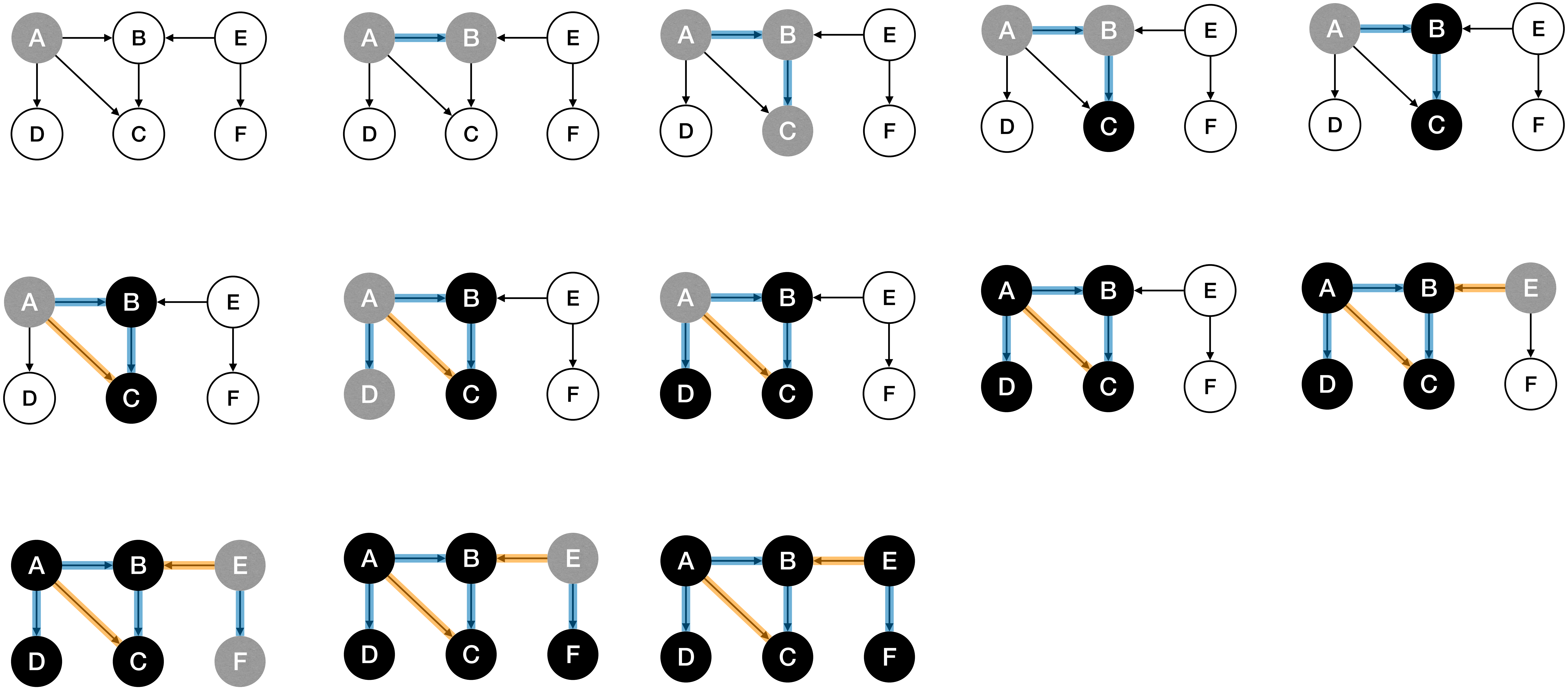
```
DFSAll(G):
for each node u in V
    v.visited := False
for each node u in V
    if !u.visited
        DFSIterSkelecton(G, u)
DFSIterSkeleton(G, s):
Stack Q
Q.push(s)
while !Q.empty()
    u := Q.pop()
    if !u.visited
        for each edge (u, v) in E
            Q.push(v)
```

Depth-First Search (DFS)

- Each node u have 3 status during DFS:
  - **Undiscovered [WHITE]**: before calling `DFSSkeleton(G,u)`
  - **Discovered [GRAY]**: during execution of `DFSSkeleton(G,u)`
  - **Finished [BLACK]**: `DFSSkeleton(G,u)` returned
- `DFS(G,u)` builds a **tree** among nodes reachable from u:
  - Root of this tree is u.
  - For each non-root, its parent is the node that makes it turn GRAY.
- DFS on entire graph builds a **forest**.

```
DFSAll(G):
for each node u in V
    u.color := WHITE
    u.parent := NIL
for each node u in V
    if u.color = WHITE
        DFS(G, u)
```

```
DFS(G, s):
s.color := GRAY
for each edge (s, v) in E
    if v.color = WHITE
        v.parent := s
        DFS(G, v)
s.color := BLACK
```


A B E D C F
A B E D C F
A B E D C F
A B E D C F
A B E D C F
A B E D C F
A B E D C F
A B E D C F
A B E D C F
A B E D C F
A B E D C F
A B E D C F
A B E D C F

Depth-First Search (DFS)

- DFS provides (at least) two chances to process each node:
  - **Pre-Visit:** WHITE → GRAY
  - **Post-Visit:** GRAY → BLACK

```
DFSAll(G):
PreProcess(G)
for each node u in V
    u.color := WHITE
    u.parent := NIL
for each node u in V
    if !u.visited
        DFS(G, u)
```

```
DFS(G, s):
PreVisit(s)
s.color := GRAY
for each edge (s, v) in E
    if v.color = WHITE
        v.parent := s
        DFS(G, v)
s.color := BLACK
PostVisit(s)
```

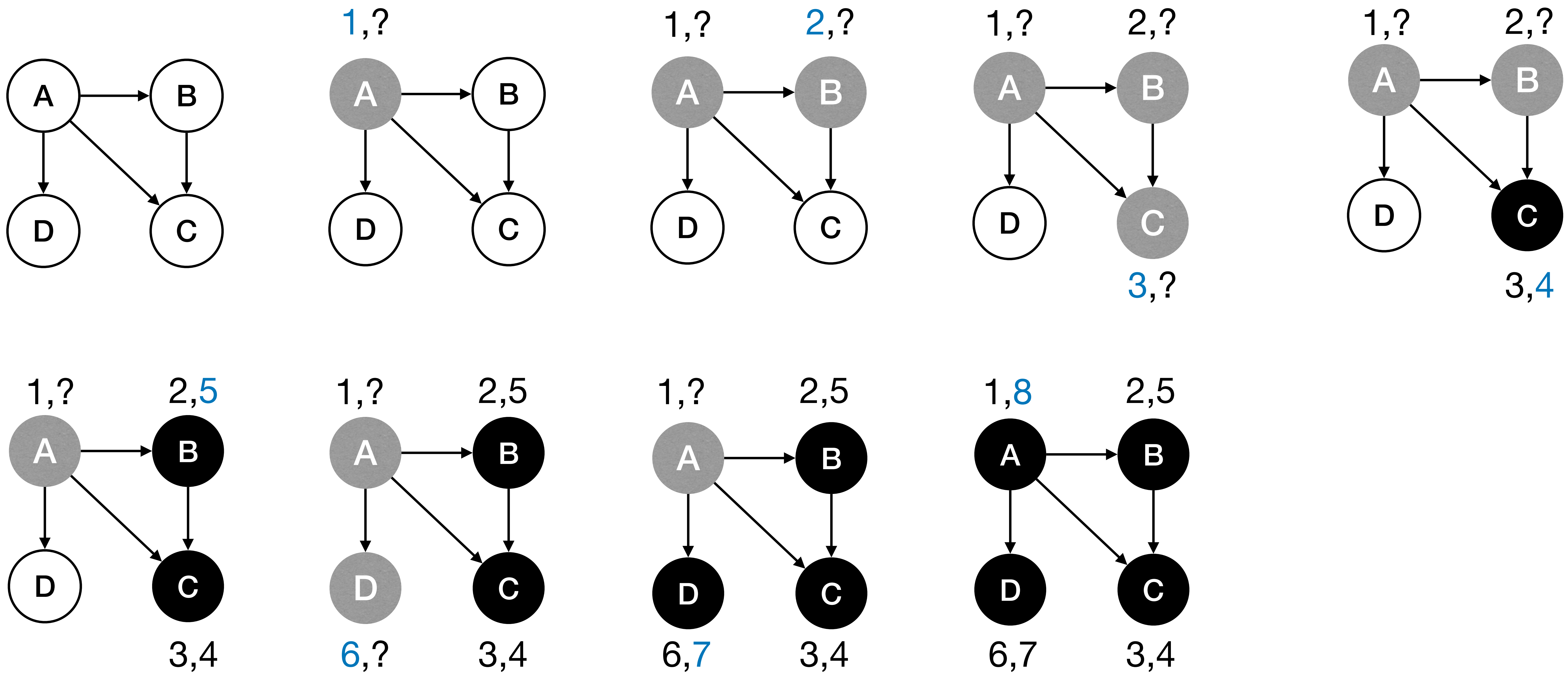
- Sample application: Track active intervals of nodes
  - Clock ticks whenever some node's color changes.
  - **Discovery time**: when the node turns to GRAY.
  - **Finish time**: when the node turns to BLACK.

```
PreProcess(G):
time := 0

PreVisit(s):
time := time + 1
s.d := time

PostProcess(G):
time := time + 1
s.f := time
```

Note: here it indicates the discovery time



DFSAll(G):

```
PreProcess(G)
for each node u in V
    u.color := WHITE
    u.parent := NIL
for each node u in V
    if !u.visited
        DFS(G, u)
```

DFS(G, s):

```
PreVisit(s)
s.color := GRAY
for each edge (s, v) in E
    if v.color = WHITE
        v.parent := s
        DFS(G, v)
s.color := BLACK
PostVisit(s)
```

PreProcess(G):

```
time := 0
```

PreVisit(s):

```
time := time + 1
s.d := time
```

PostProcess(G):

```
time := time + 1
s.f := time
```

Runtime of DFS

- Time spent on each node: $O(1)$
  - `DFS(G,u)` is called once for each node u.
- Time spent on each edge: $O(1)$
  - Each edge is examined $O(1)$ times.

Total runtime: $O(n+m)$

DFSAll(G):

PreProcess(G)
for each node u **in** V
  $u.color :=$ *WHITE*
  $u.parent := NIL$
for each node u **in** V
  if $!u.visited$
    $DFS(G, u)$

DFS(G, s):

PreVisit(s)
$s.color := GRAY$
for each edge (s, v) **in** E
  if $v.color =$ WHITE
    $v.parent := s$
    $DFS(G, v)$
$s.color := BLACK$
PostVisit(s)

PreProcess(G):

$time := 0$

PreVisit(s):

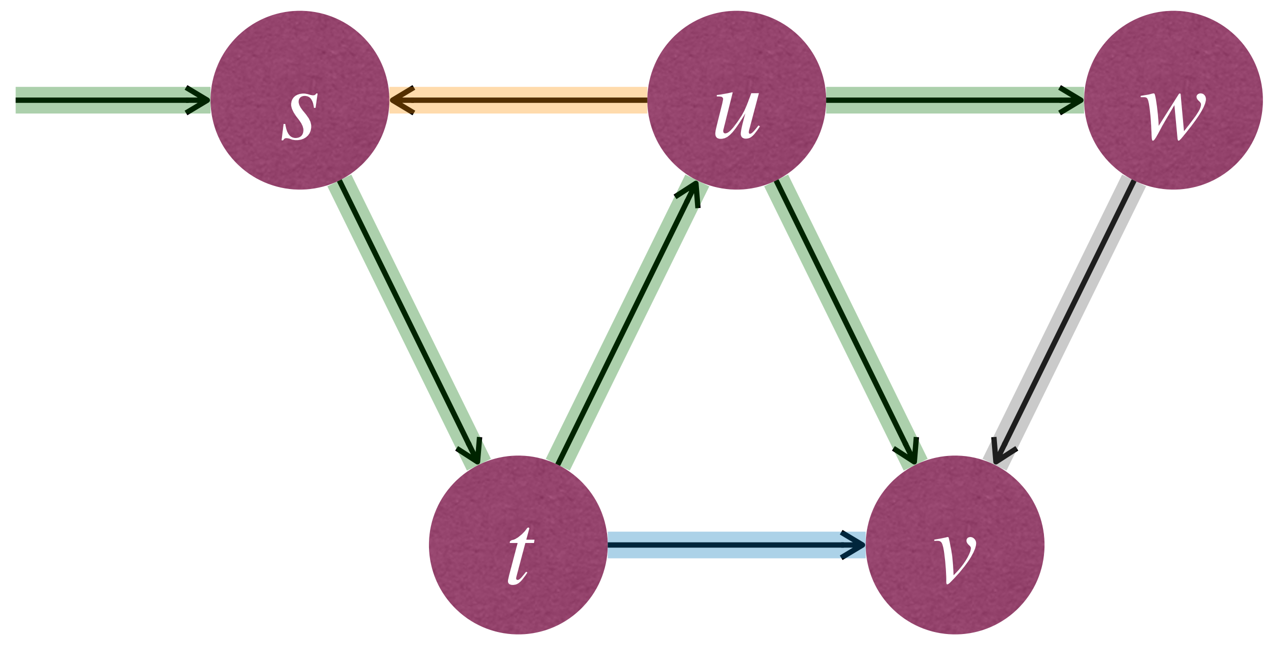
$time := time + 1$
$s.d := time$

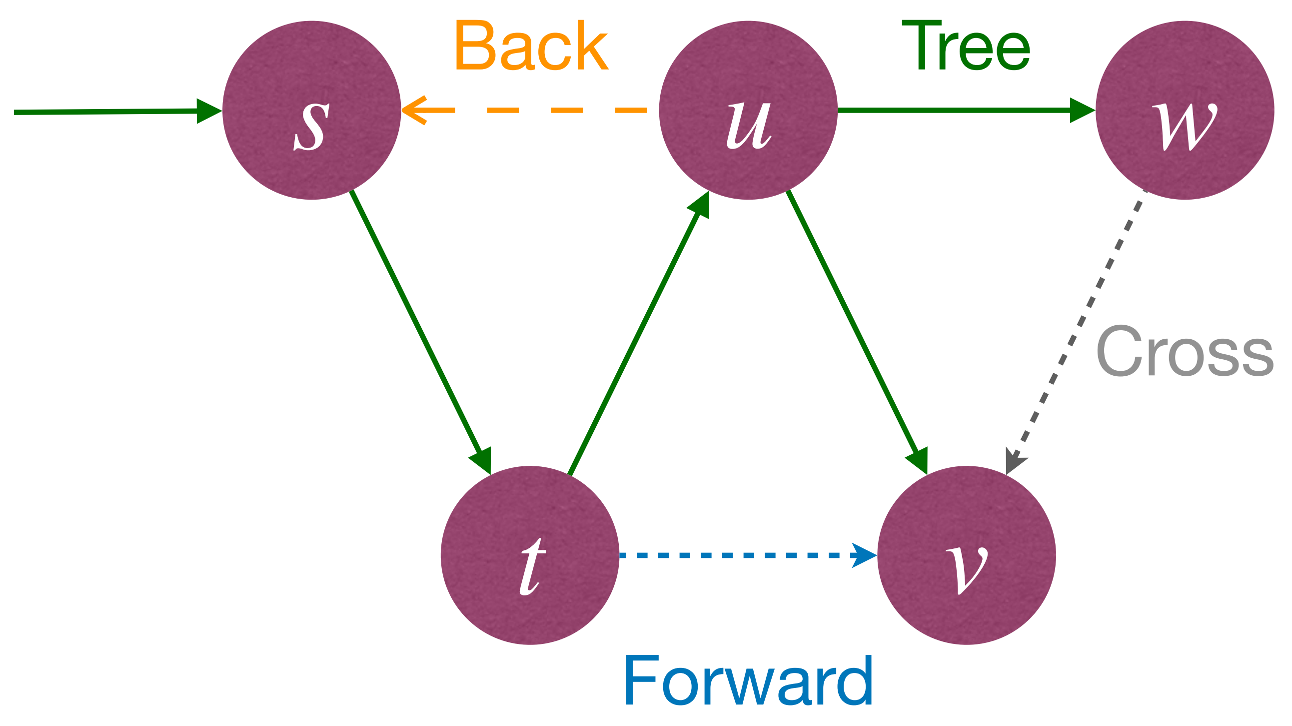
PostProcess(G):

$time := time + 1$
$s.f := time$

Classification of edges

- DFS process classify edges of input graph into four types.
  - **Tree Edges**: Edges in the DFS forest.
  - **Back Edges**: Edges (u, v) connecting u to an ancestor v in a DFS tree.
  - **Forward Edges**: Non-tree edges (u, v) connecting u to a descendant v in a DFS tree.
  - **Cross Edges**: Other edges. (Connecting nodes in same DFS tree with no ancestor-descendant relation, or connecting nodes in different DFS trees.)



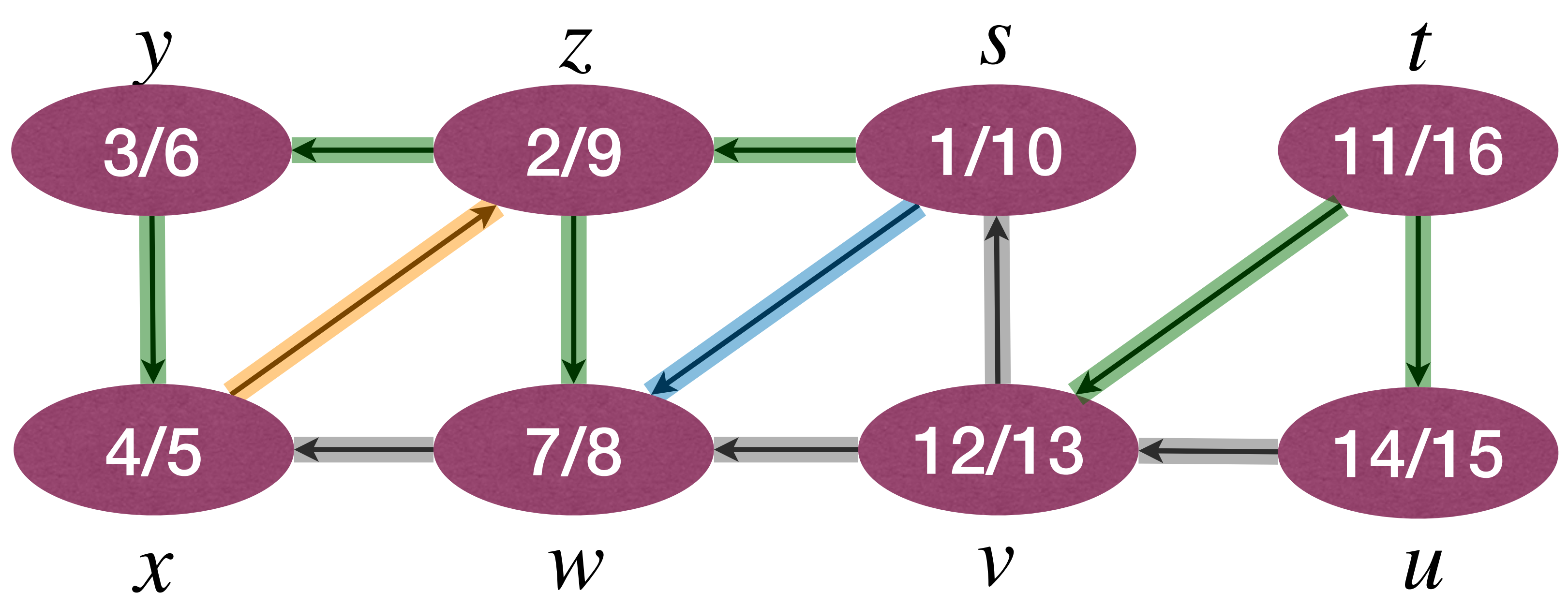


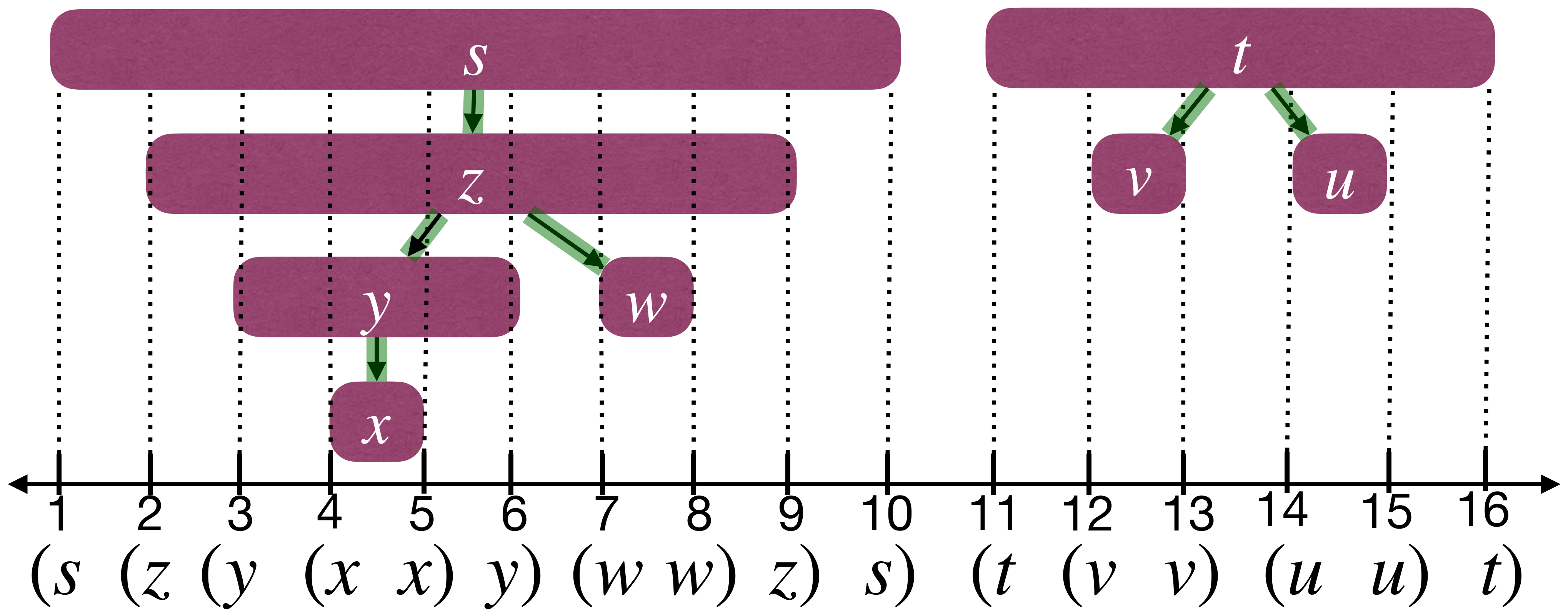
Tree Back Forward Cross

Properties of DFS: Parenthesis Theorem

Theorem: Active intervals of two nodes are either: (**a**) entirely disjoint; or (**b**) one is entirely contained within another.

- For any two nodes u and v, *exactly* one of following holds:
  - (**a**) $[u.d, u.f\,]$ and $[v.d, v.f\,]$ are disjoint, and u, v have no ancestor-descendant relation in the DFS forest;
  - (**b**) $[u.d, u.f\,] \subset [v.d, v.f\,]$, and u is a descendant of v in a DFS tree;
  - (**c**) $[v.d, v.f\,] \subset [u.d, u.f\,]$, and u is an ancestor of v in a DFS tree.





Properties of DFS: Parenthesis Theorem

- **Proof**: Consider two nodes u and v. W.l.o.g., assume $u.d < v.d$.

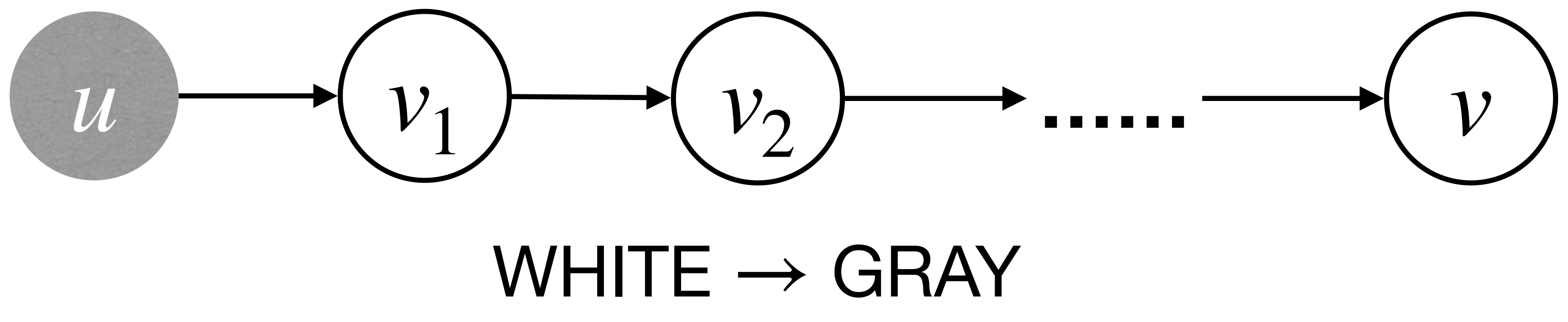
- If $v.d < u.f$, then v is discovered (WHITE → GRAY) while u is being processed (GRAY); and DFS will finish v first, before returning to u.
  - In this case, $[v.d, v.f\,] \subset [u.d, u.f\,]$, and u is an ancestor of v.

- If $v.d > u.f$, then obviously $u.d < u.f < v.d < v.f$; and DFS has finished exploring u (BLACK), before v is discovered (WHITE → GRAY).
  - In this case, $[u.d, u.f\,]$ and $[v.d, v.f\,]$ are disjoint, and u, v have no ancestor-descendant relation.

Properties of DFS: White-path Theorem

Theorem In the DFS forest, v is a descendant of u iff when u is discovered, there is a path in the graph from u to v containing only WHITE nodes.

- Proof of [$\Longrightarrow$]
  - **Claim**: If v is a proper descendant of u, then v is WHITE when u is discovered.
    - Since if v is a is a proper descendant of u, then $u.d < v.d$.
  - For any node along the path from u to v in the DFS forest, above claim holds.
  - Therefore, [$\Longrightarrow$] direction of the theorem holds.

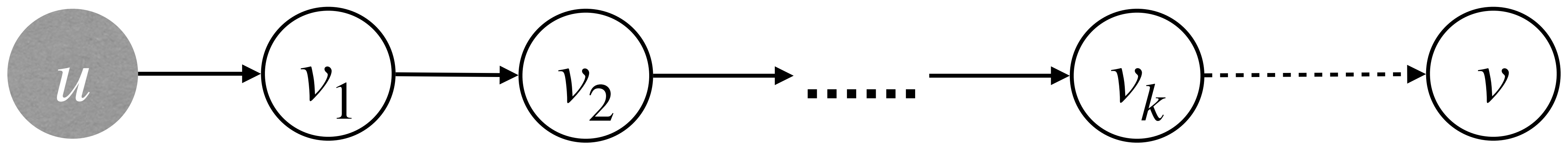


Properties of DFS: White-path Theorem

- Proof of [$\Longleftarrow$]:
  - W.l.o.g., assume v is the *first* node along the path that does *not* become a descendant of u.
  - So we have $[v_k.d, v_k.f\,] \subset [u.d, u.f\,]$.
  - But v is discovered after u is discovered, and must before v_k is finished.

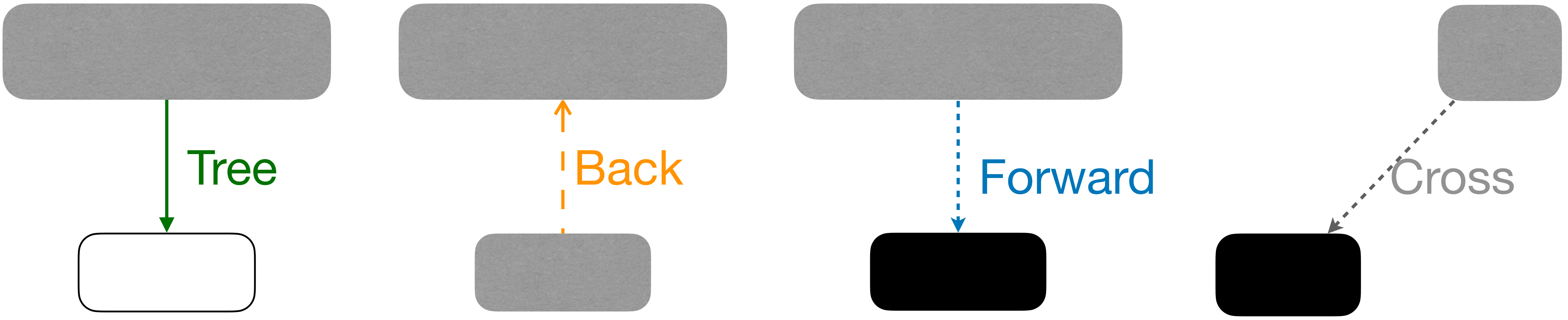
Depth-first search until all the edges of v_k is explored!

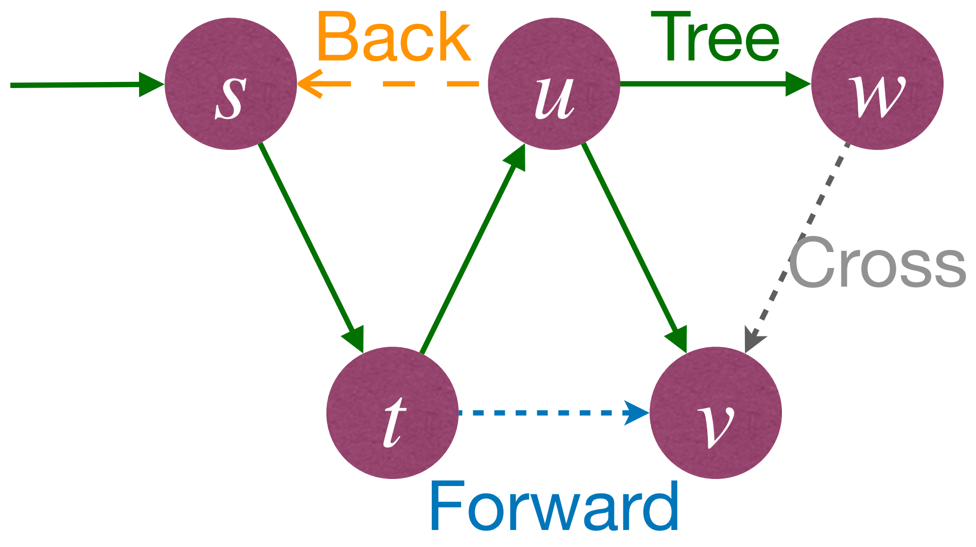
  - So we have $u.d < v.d < v_k.f \le u.f$.
  - Then it must be $[v.d, v.f\,] \subset [u.d, u.f\,]$, implying v is a descendant of u.

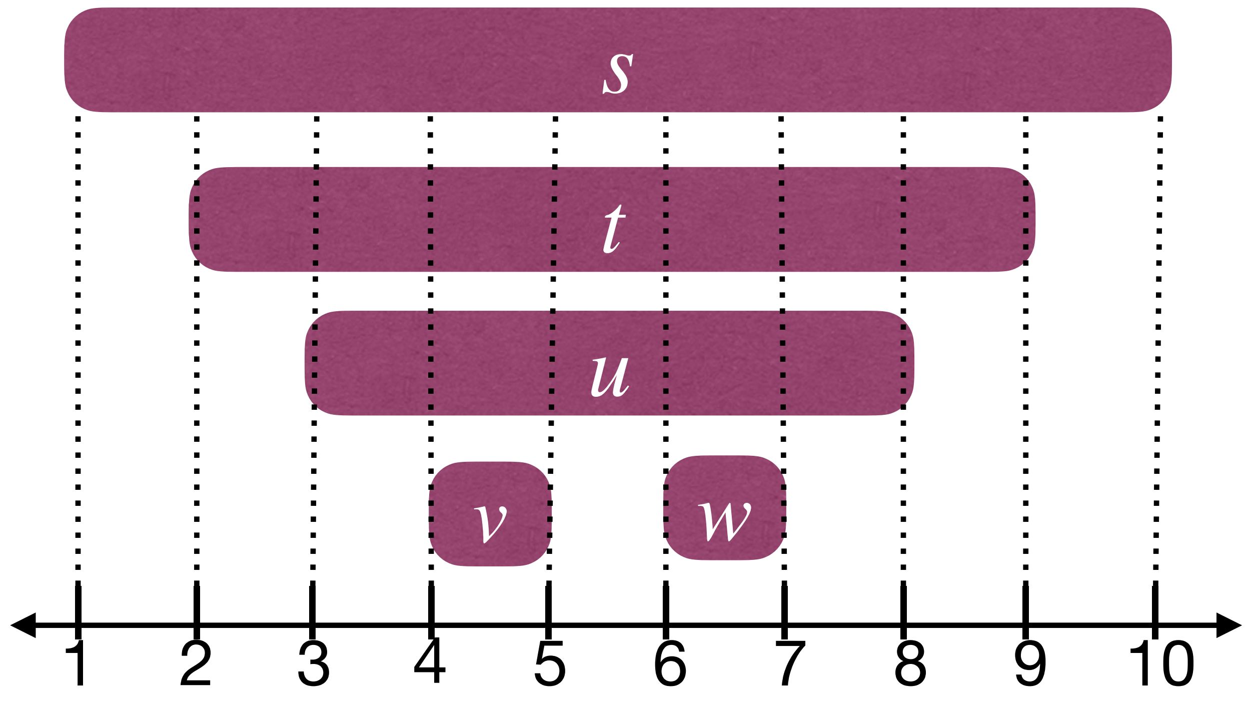


Properties of DFS: Classification of edges

- Determine (u, v) type by color of v during DFS execution.
  - **Tree Edges**: Edges in the DFS forest. — Node v is WHITE
  - **Back Edges**: Edges (u, v) connecting u to an ancestor v in a DFS tree. — Node v is GRAY
  - **Forward Edges**: Non-tree edges (u, v) connecting u to a descendant v in a DFS tree. — Node v is BLACK
  - **Cross Edges**: Other edges. (Connecting nodes in same DFS tree with no ancestor-descendant relation, or connecting nodes in different DFS trees.) — Node v is BLACK







Properties of DFS: Types of edges in undirected graphs

- Will all four types of edges appear in DFS of undirected graphs?

> Theorem In DFS of an *undirected* graph G, every edge of G is either a ***tree edge*** or a ***back edge***.

- **Proof:**
  - Consider an arbitrary edge (u, v). W.l.o.g., assume $u.d < v.d$.
  - Edge (u, v) must be explored while u is GRAY. WHY?
  - Consider the first time the edge (u, v) is explored.

Properties of DFS: Types of edges in undirected graphs

- Proof (continued):
  - If the direction is $u \rightarrow v$. Then, v must be WHITE by then, for otherwise the edge would have been explored from direction $v \rightarrow u$ earlier.
    - In such case, the edge (u, v) becomes a **tree edge.**
  - If the direction is $v \rightarrow u$. Then, the edge is "GRAY → GRAY".
    - In such case, the edge (u, v) becomes a **back edge.**

DFS, BFS, and others…

DFSIterSkeleton(G, s):

```
Stack Q
Q.push(s)
while !Q.empty()
    u := Q.pop()
    if !u.visited
        u.visited := True
        for each edge (u, v) in E
            Q.push(v)
```

BFSSkeletonAlt(G, s):

```
FIFOQueue Q
Q.enque(s)
while !Q.empty()
    u := Q.dequeue()
    if !u.visited
        u.visited := True
        for each edge (u, v) in E
            Q.enque(v)
```

GraphExploreSkeleton(G, s):

```
GenericQueue Q
Q.add(s)
while !Q.empty()
    u := Q.remove()
    if !u.visited
        u.visited := True
        for each edge (u, v) in E
            Q.add(v)
```

Other queuing disciplines lead to more interesting algorithms!

Further reading

- [CLRS] Ch.22 (22.1-22.3)