深度优先的一些应用
Some application of DFS

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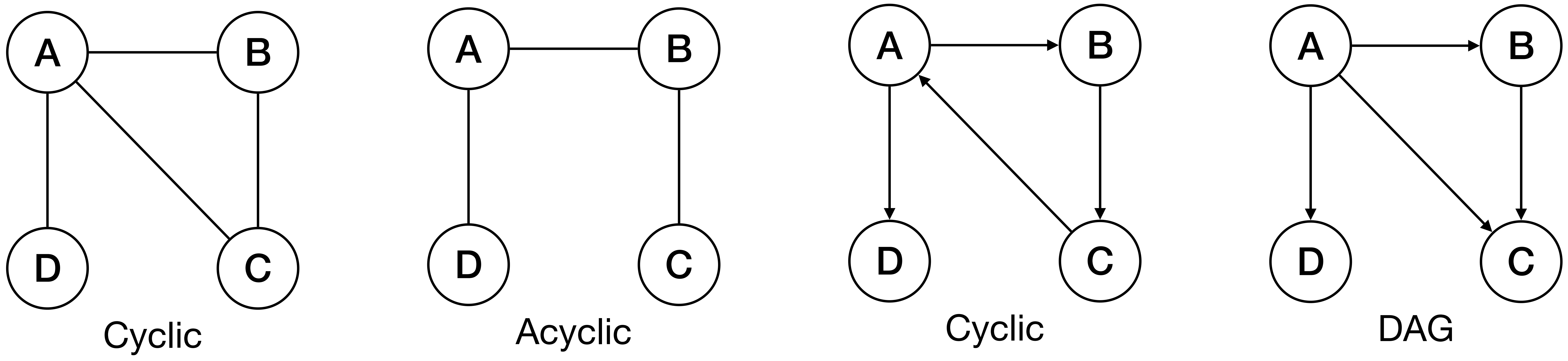
The slides are mainly adapted from the original ones shared by Chaodong Zheng and Kevin Wayne. Thanks for their supports!

Directed Acyclic Graphs (DAG)

- A graph without cycles is called acyclic.
- A directed graph without cycles is a directed acyclic graph (DAG).



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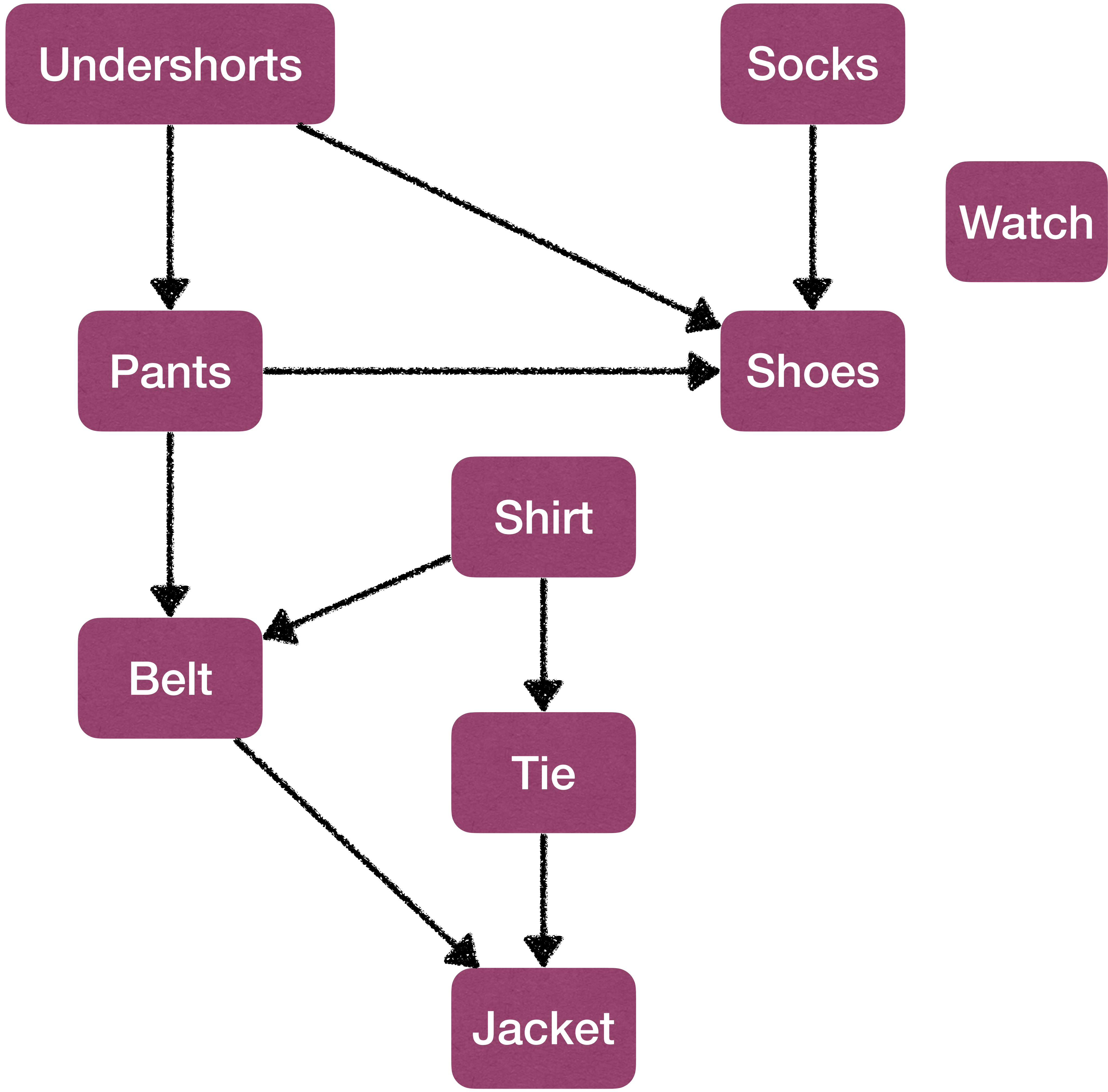
Application of DAG

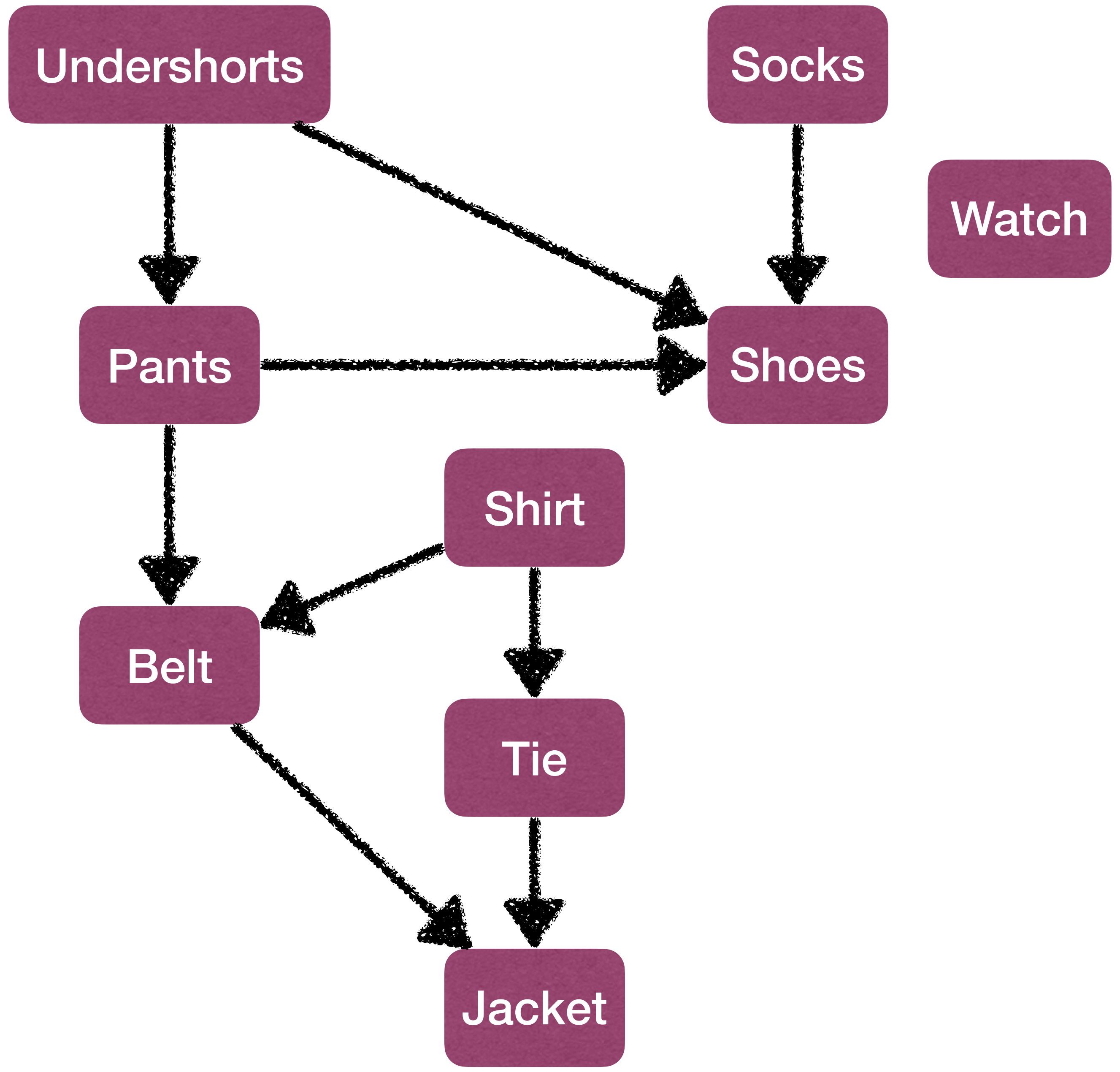
- DAGs are good for modeling relations such as: causalities, hierarchies, and temporal dependencies.
- For example:
  - Consider how you get dressed in the morning.
    - Must wear certain garments before others (e.g., socks before shoes).
    - Other items may be put on in any order (e.g., socks and pants).
  - This process can be modeled by a DAG!

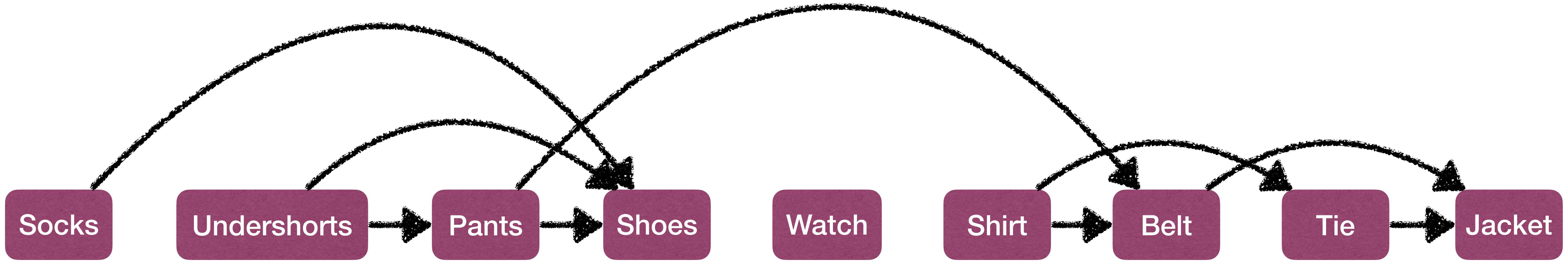


What is a valid order to perform all the task?

Topological Sort

- A **topological sort** of a DAG G is a linear ordering of its vertices such that if G contains an edge (u, v) then u appears before v in the ordering.
- $E(G)$ defines a **partial order** over $V(G)$, a **topological sort** gives a **total order** over $V(G)$ satisfying $E(G)$



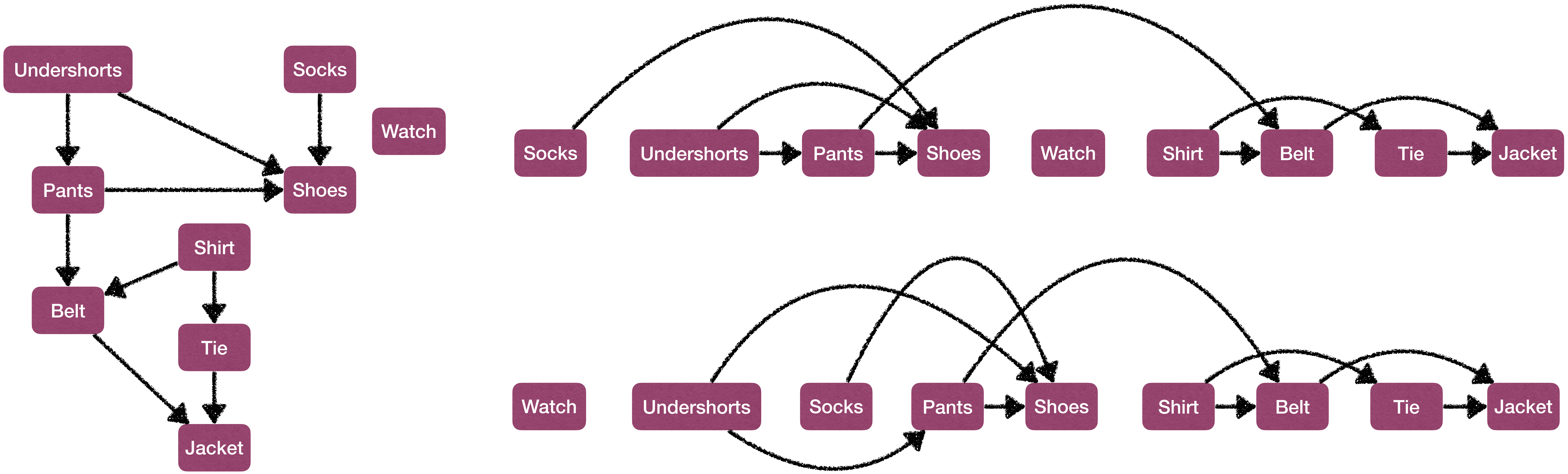


A topological ordering arranges the vertices along a horizontal line so that all edges go "from left to right".

Topological Sort

- **Topological sort** is **impossible** if the graph contains a cycle.
- A given graph may have multiple different valid topological ordering.

How to generate a topological ordering?



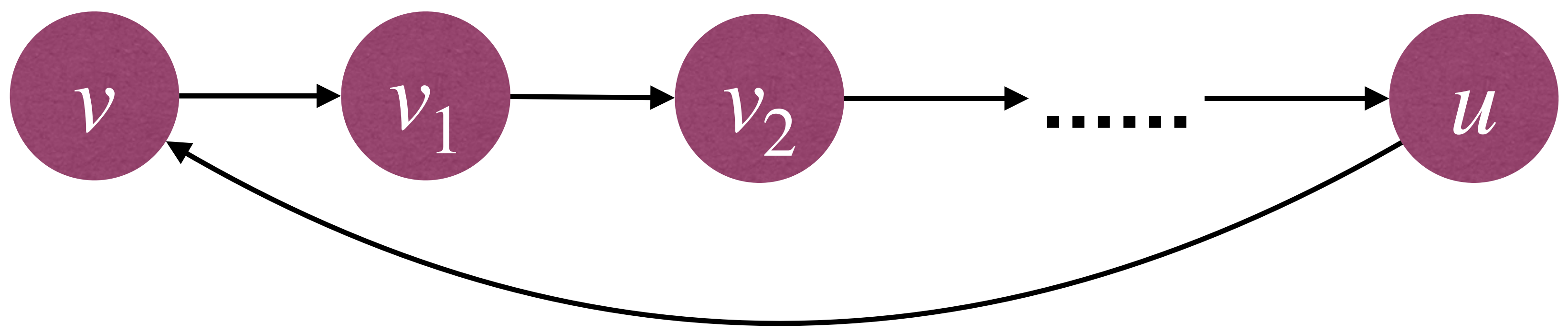
Topological Sort

- A topological sort of a DAG G is a linear ordering of its vertices such that if G contains an edge (u, v) then u appears before v in the ordering.
- **Question**: Does ***every*** DAG has a topological ordering?
- **Question**: How to tell if a directed graph is acyclic?
  - And if acyclic, how to do topological sort?

Topological Sort

Lemma 1 Directed graph G is acyclic iff a DFS of G yields no **back** edges

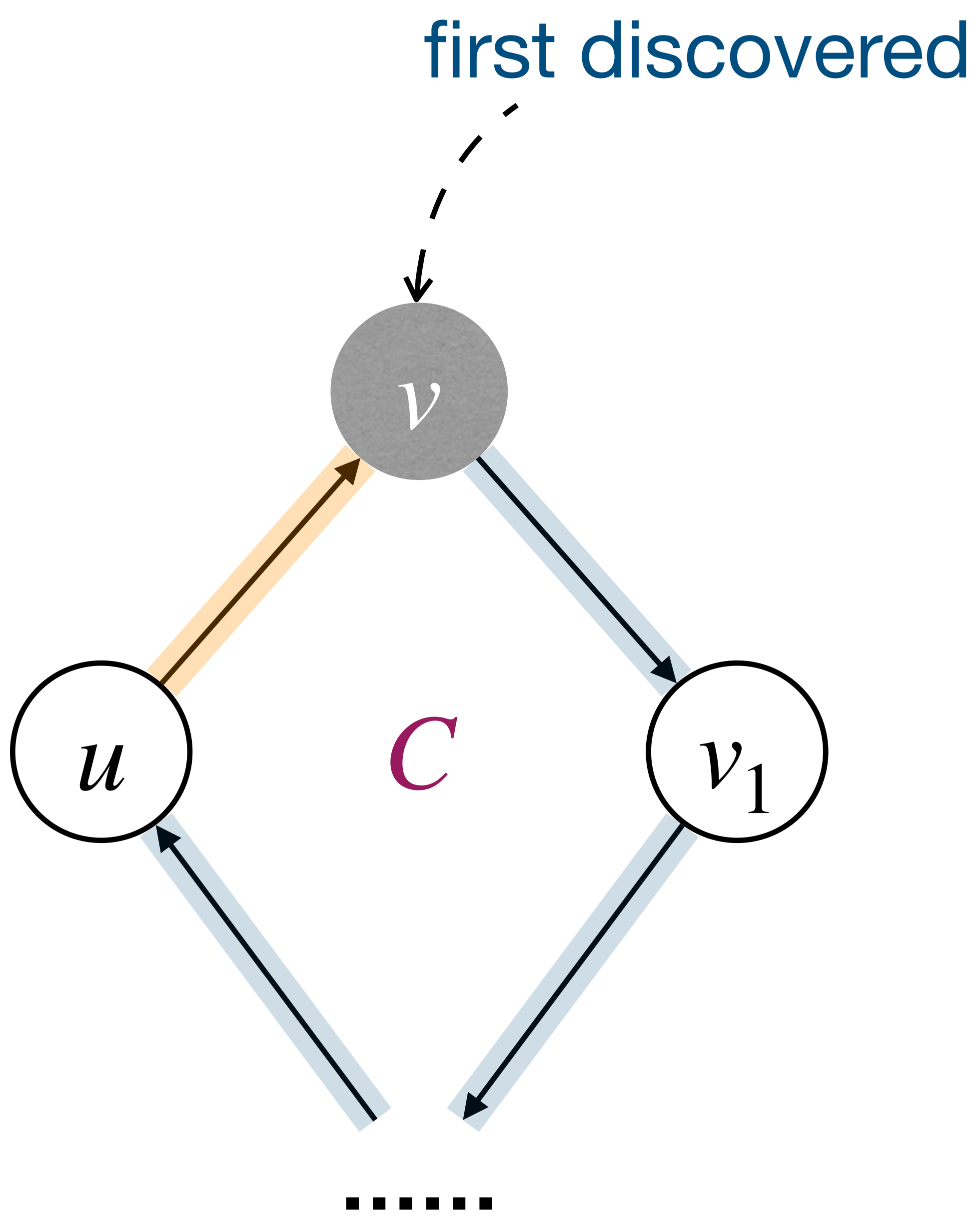
- Proof of [$\Longrightarrow$] (Directed graph G is acyclic $\Longrightarrow$ a DFS of G yields no back edges)
  - For the sake of contradiction, assume DFS yields back edge (u, v).
  - So v is ancestor of u in DFS forest, meaning there's a path from v to u in G.
  - But together with edge (u, v) this creates a cycle. Contradiction!



Topological Sort

Lemma 1 Directed graph G is acyclic iff a DFS of G yields no **back** edges

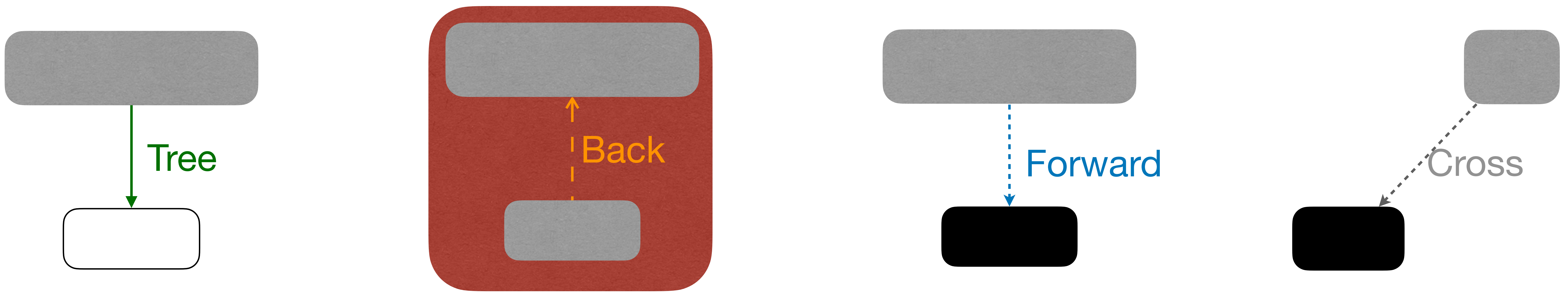
- Proof of [$\Longleftarrow$] (Directed graph G is acyclic $\Longleftarrow$ a DFS of G yields no back edges)
  - For the sake of contradiction, assume G contains a cycle C.
  - Let v be the first node to be discovered in C.
  - By the **White-path** theorem, u is a descendant of v in DFS forest.
  - But then when processing u, (u, v) becomes a back edge!



Topological Sort

Lemma 2 If we do a DFS in DAG G, then $u.f > v.f$ for every edge (u,v) in G

- Proof:
  - When exploring (u, v), v cannot be GRAY. (Otherwise we have a back edge.)
  - If v is WHITE, then v becomes a descendant of u, and $u.f > v.f$
  - If v is BLACK, then trivially $u.f > v.f$



Topological Sort

- A topological sort of a DAG G is a linear ordering of its vertices such that if G contains an edge (u, v) then u appears before v in the ordering.
- **Q**: Does every DAG has a topological ordering?
- **Q**: How to tell if a directed graph is acyclic? If acyclic, how to do topological sort?

Lemma 1 Directed graph G is acyclic iff a DFS of G yields no back edges

Lemma 2 If we do a DFS in DAG G, then $u.f > v.f$ for every edge (u,v) in G

Theorem Decreasing order of finish times of DFS on DAG gives a topological ordering

Corollary Every DAG has a topological ordering

Topological Sort

- Topological Sort of G:

  (a) Do DFS on G, compute finish times for each node along the way.

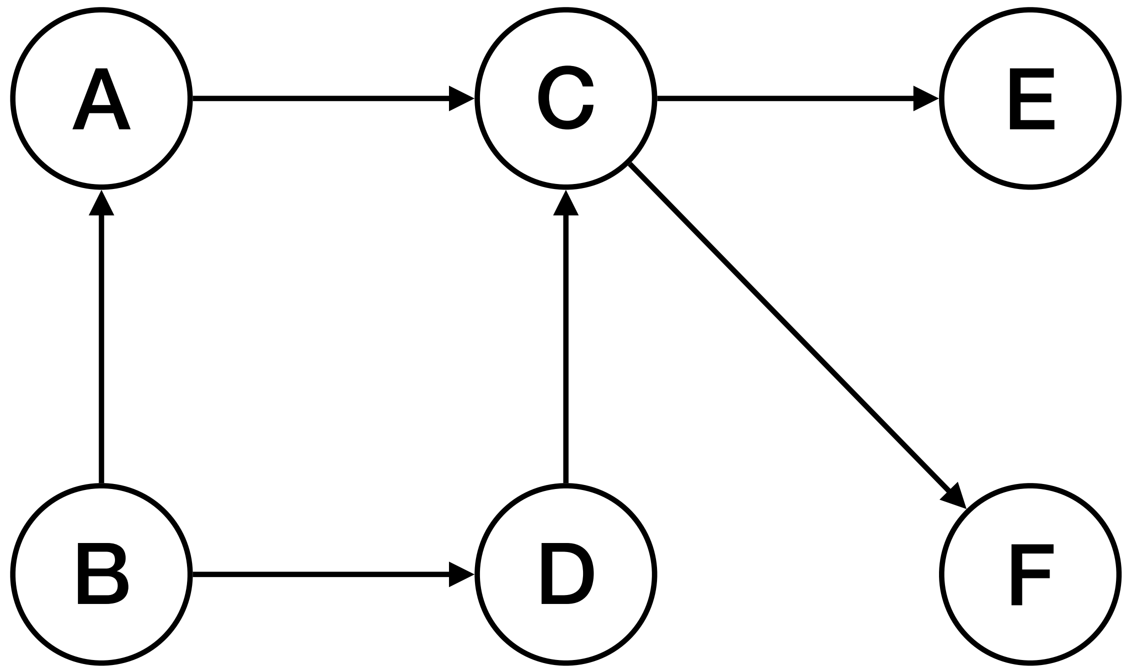
  (b) When a node finishes, insert it to the head of a list.

  (c) If no back edge is found, then the list eventually gives a Topological Ordering.

Time complexity is $O(n+m)$

Topological Sort

Source and Sink in DAG



- A **source node** is a node with no incoming edges;
- A **sink node** is a node with no outgoing edges.
  - Example: B is source; both E and F are sink.
- **Claim**: Each DAG has at least one source and one sink. WHY?
- **Observations**: In DFS of a DAG, node with max finish time must be a source
  - Node with max finish time appears first in topological sort, it cannot have incoming edges.
- **Observations**: In DFS of a DAG, node with min finish time must be a sink.
  - Node with min finish time appears last in topological sort, it cannot have outgoing edges.

Alternative Algorithm for Topological Sort

(1) Find a source node s in the (remaining) graph, output it.

(2) Delete s and all its outgoing edges from the graph.

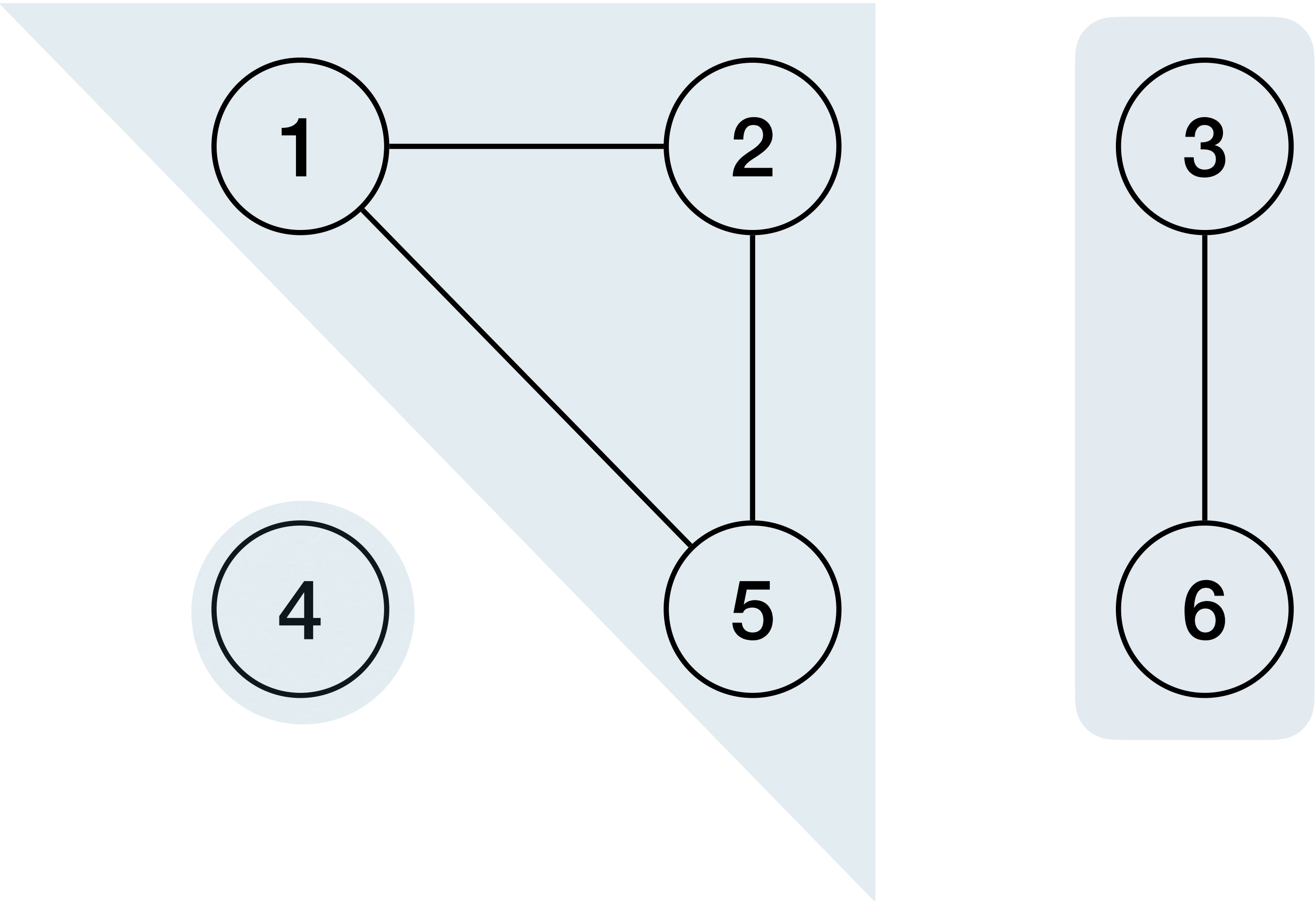
(3) Repeat until the graph is empty.

Formal proof of correctness?
How efficient can you implement it?

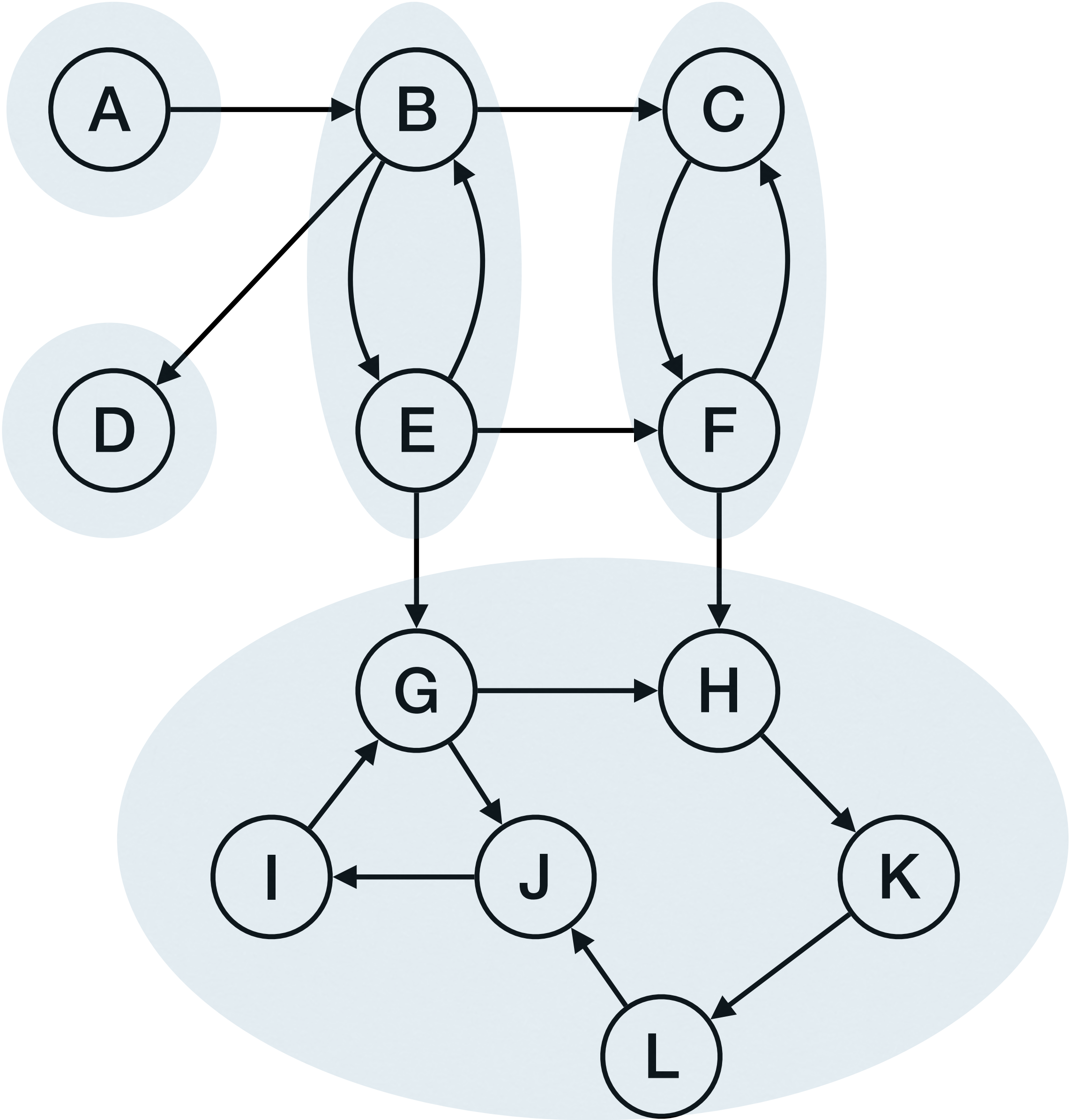
(Strongly) Connected Components

(Strongly) Connected Components

- For an **undirected** graph G, a **Connected Component (CC)** is a **maximal** set $C \subseteq V(G)$, such that for any pair of nodes u and v in C, there is a path from u to v.
  - E.g.: {4}, {1, 2, 5}, {3, 6}



- For a **directed** graph G, a **Strongly Connected Component (SCC)** is a **maximal** set $C \subseteq V(G)$, such that for any pair of nodes u and v in C, there is a **directed** path from u to v, and **vice versa**.
  - **E.g.**: $\{A\}, \{D\}, \{B, E\}, \{C, F\}, \{G, H, I, J, K, L\}$

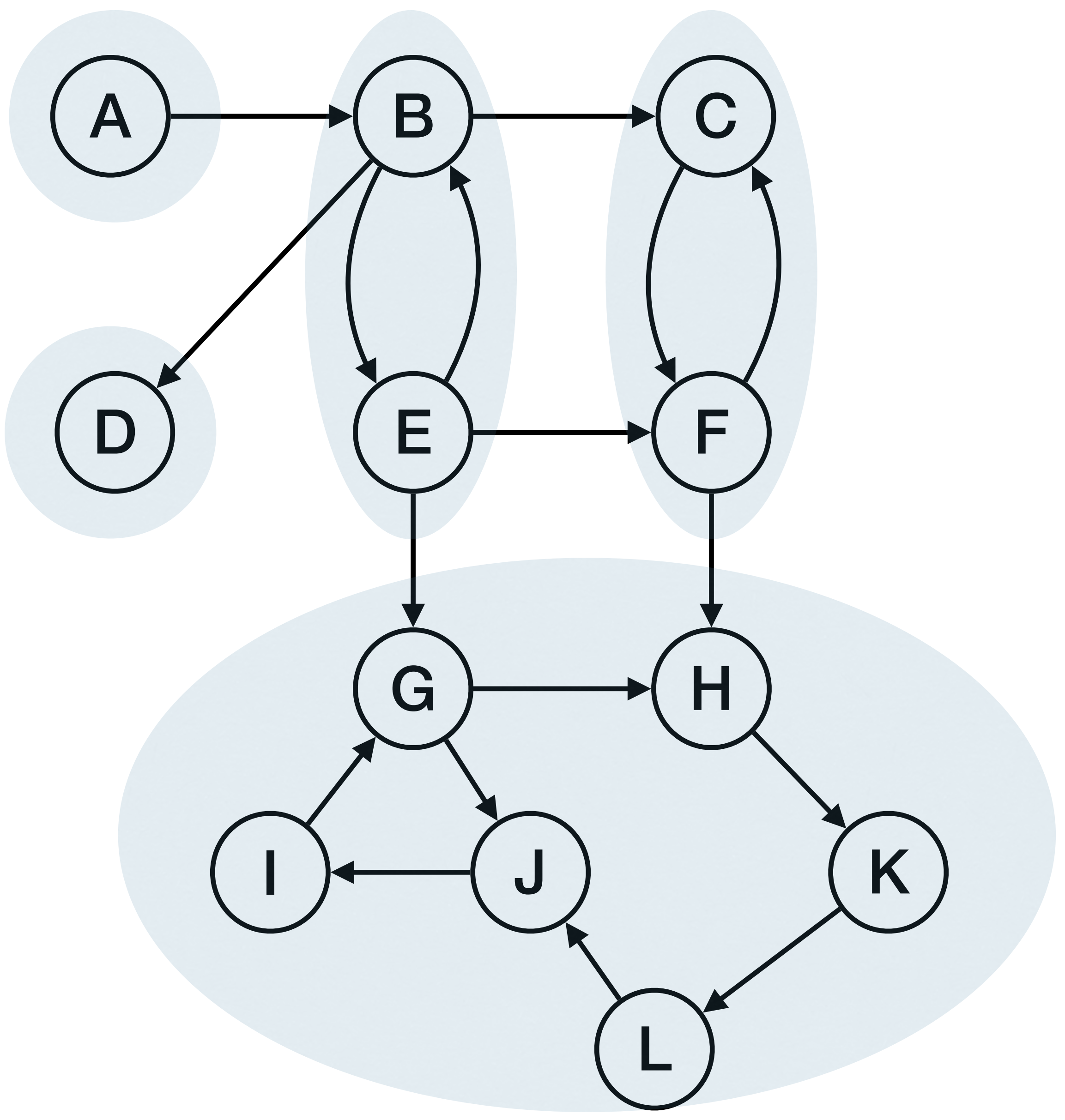


Computing CC and SCC

- Given an undirected graph, how to compute its connected components (CC) ?
  - Easy, just do DFS (or BFS) on the entire graph.
    - DFS(u) (or BFS(u)), reaches exactly nodes in the CC containing u.
- Given a directed graph, how to compute its strongly connected components (SCC) ?
  - Err, can be done efficiently, but not so obvious…

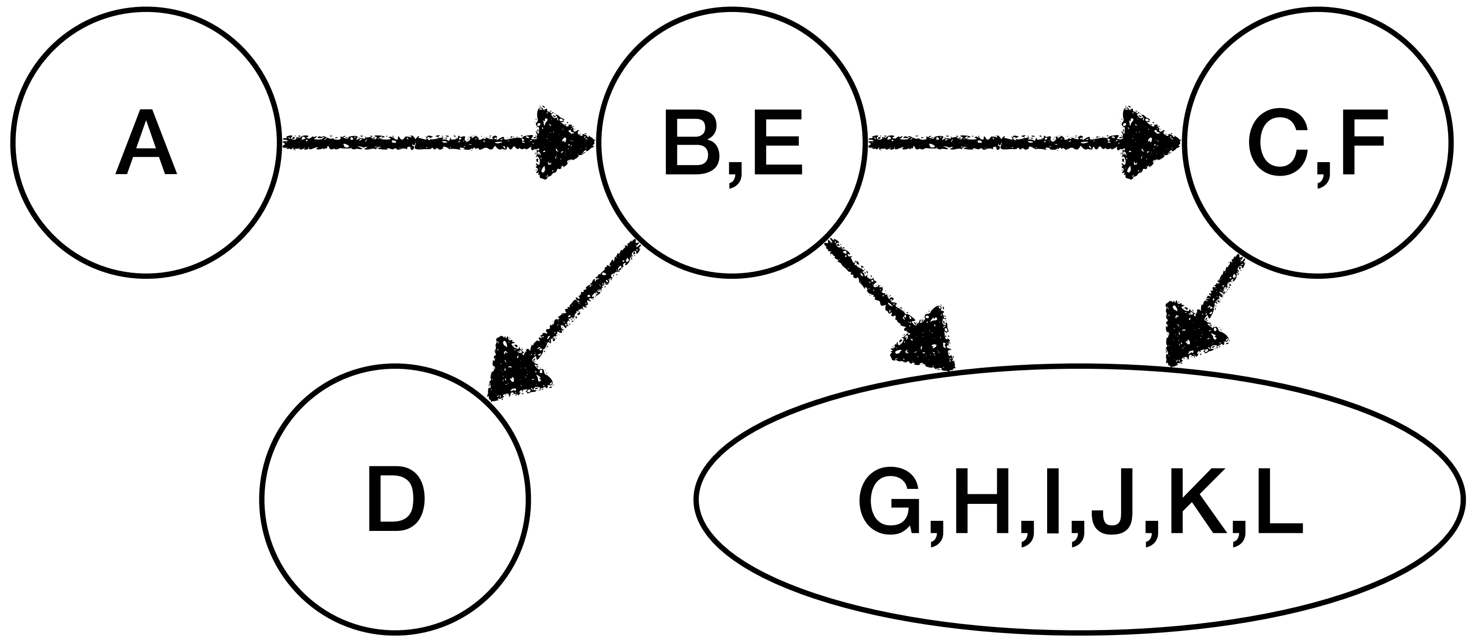
Component Graph

- Given a directed graph $G = (V, E)$, assume it has k SCC $\{C_1, C_2, \ldots, C_k\}$, then the **component graph** is $G^C = (V^C, E^C)$.
  - The vertex set V^C is $\{v_1, v_2, \ldots, v_k\}$, each representing one SCC.
  - There is an edge $(v_i, v_j) \in E^C$ if there exists $(u, v) \in E$, where $u \in C_i$ and $v \in C_j$.



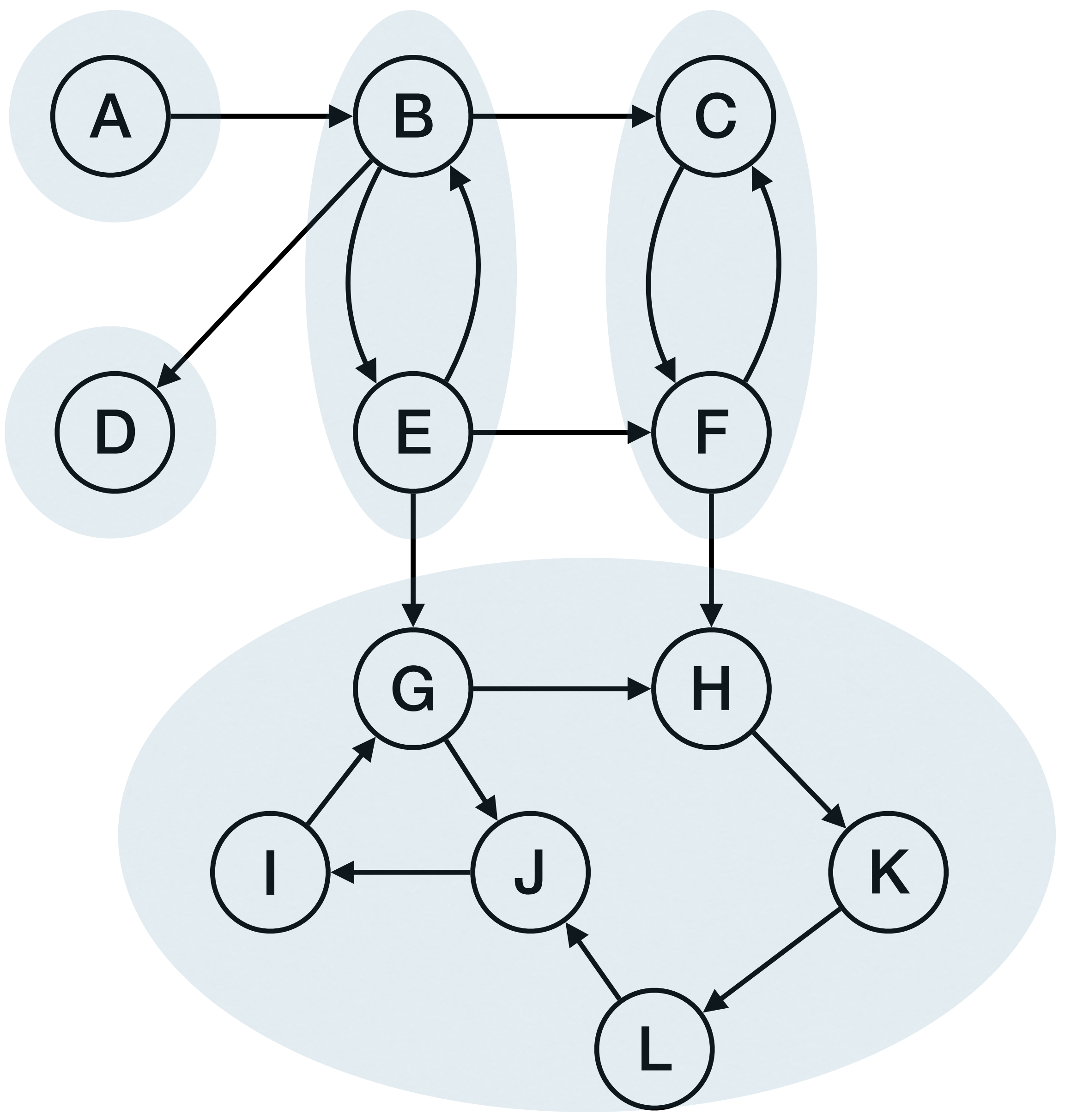
Claim: A component graph is a DAG!

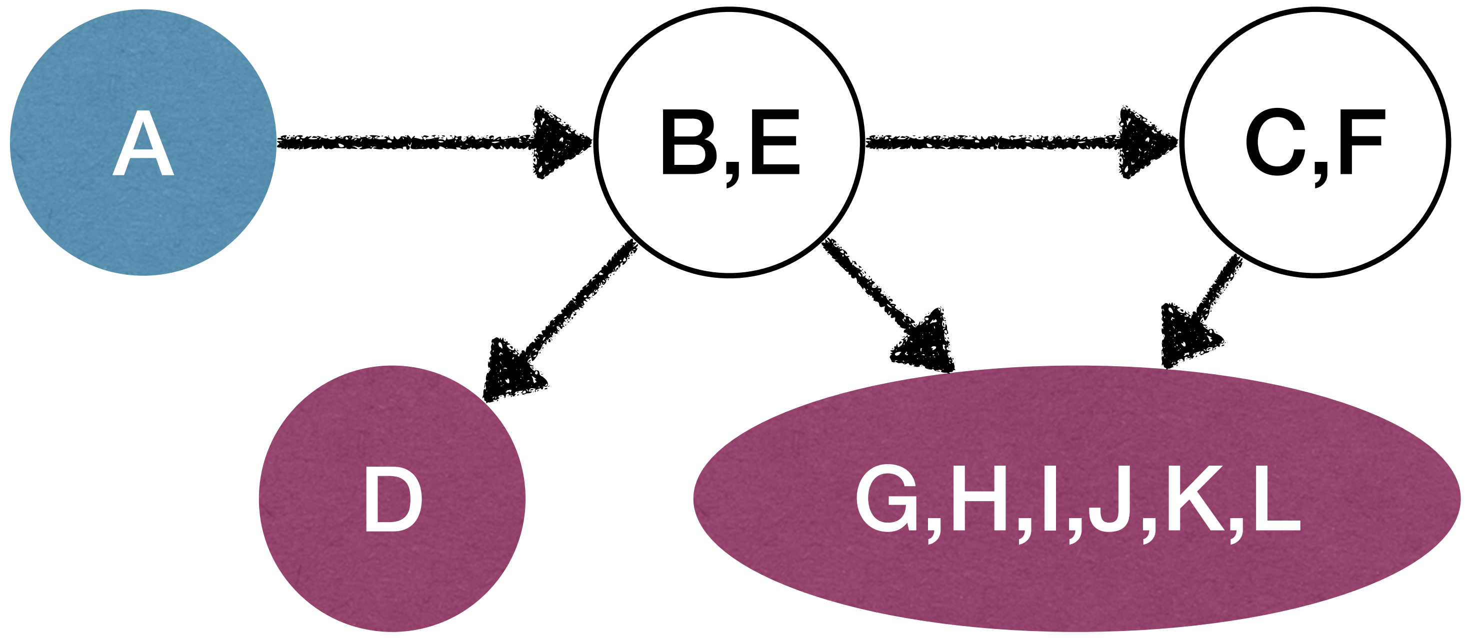
- **Proof:** Otherwise, the components in the circle becomes a bigger SCC, contradiction!



Computing SCC

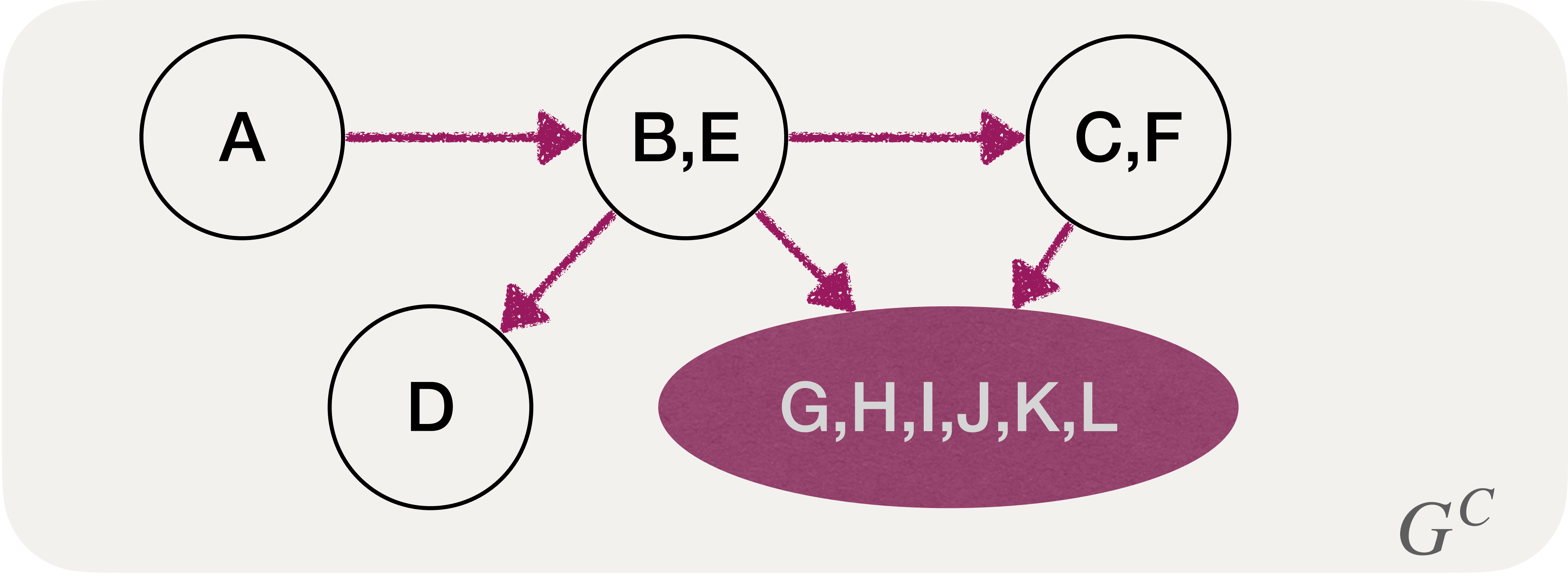
- A component graph is a DAG.
- Each DAG has at least one **source** and one **sink**.
- If we do one DFS starting from a node in a sink SCC, then we explore exactly nodes in that SCC and stop!
  - Due to the white-path theorem.
- A good start, but two problems exist:
  - (1) How to identify a node that is in a sink SCC?
  - (2) What to do when the first SCC is done?

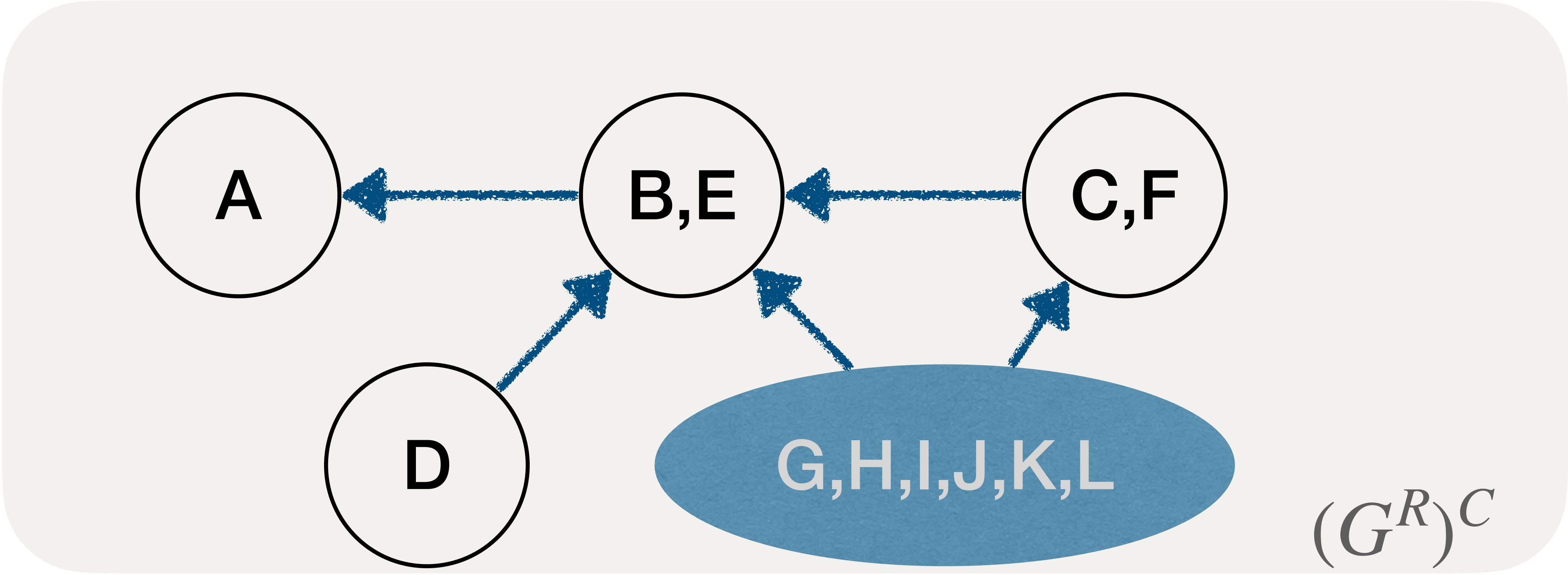




Computing SCC

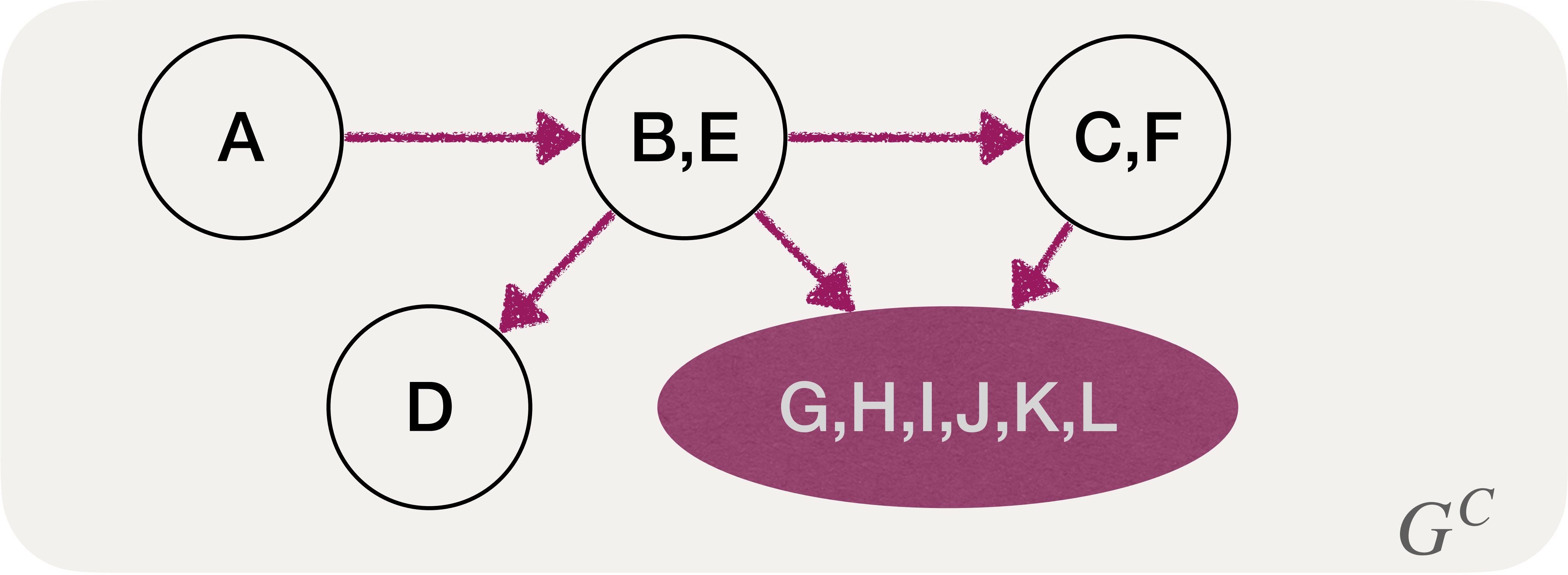
- (1) How to identify a node that is in a sink SCC?
- (2) What to do when the first SCC is done?
- Don't do it directly: find a node in a *source* SCC!
- Reverse the direction of each edge in G gets G^R.
- G and G^R have the same set of SCCs.
- G^C and $(G^R)^C$ have same vertex set, but the direction of each edge is reversed.
- A source SCC in $(G^R)^C$ is a sink SCC in G^C.

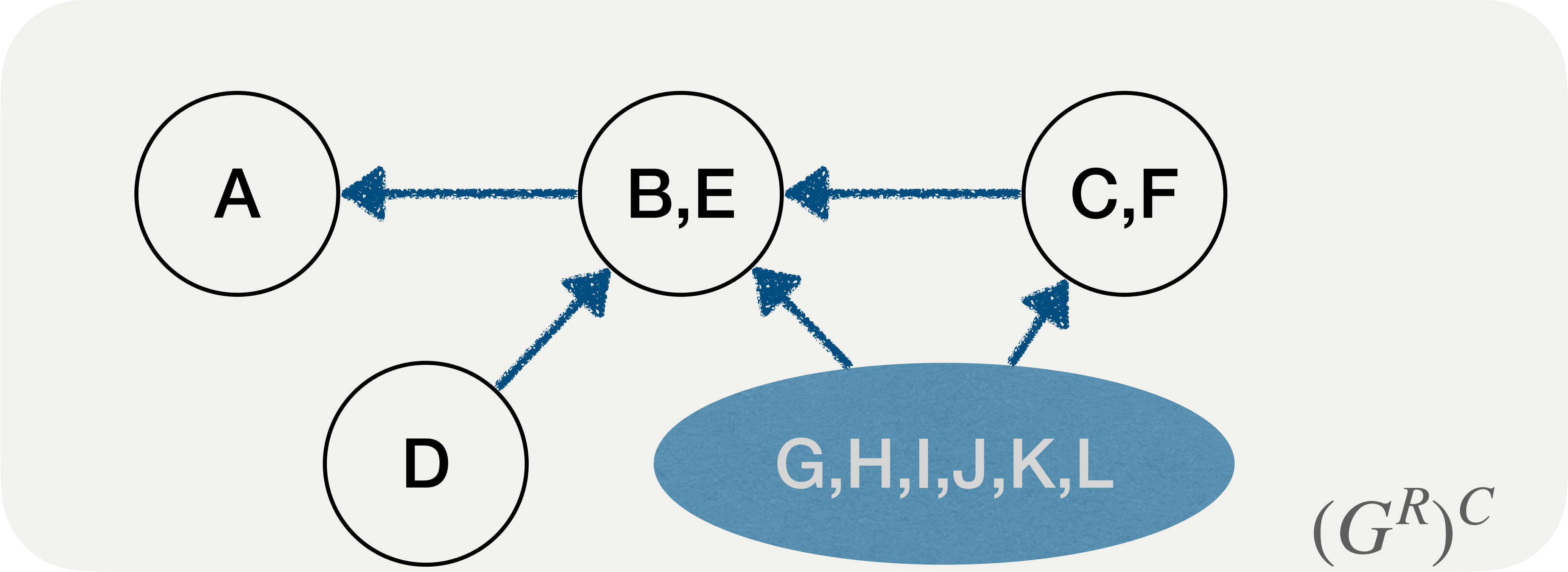




Computing SCC

- (1) How to identify a node that is in a sink SCC?
- (2) What to do when the first SCC is done?
- Compute G^R in $O(n+m)$ time, then find a node is a source SCC in G^R!
- But how to find such a node?
    - Do DFS in G^R, the node with maximum finish time is guaranteed to be in source SCC.

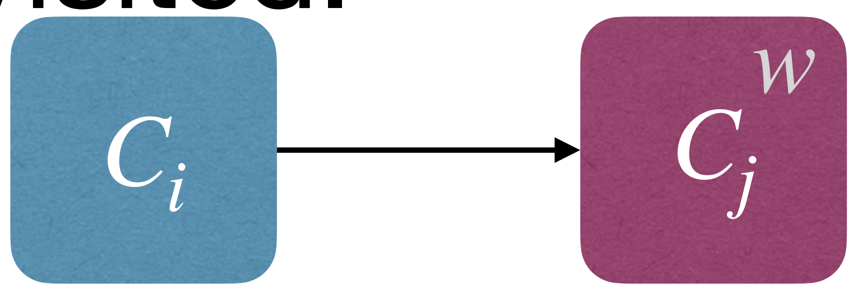




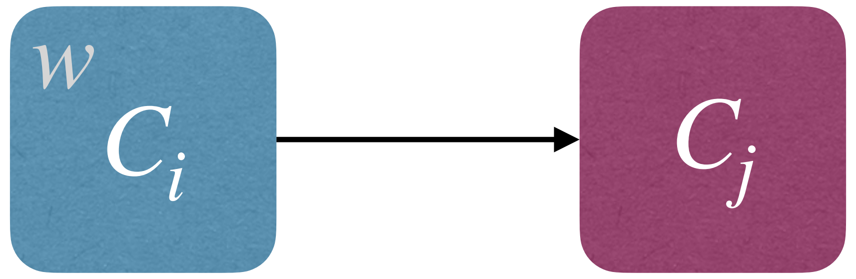
Computing SCC

Lemma For any edge $(u, v) \in E(G^R)$, if $u \in C_i$ and $v \in C_j$, then $\max_{u \in C_i}\{u.f\} > \max_{v \in C_j}\{v.f\}$

- Proof:
  - Consider nodes in C_i and C_j, let w be the first node visited by DFS.
  - If $w \in C_j$, then all nodes in C_j will be visited before any node in C_i is visited.



  - In this case, the lemma clearly is true.
  - If $w \in C_i$, at the time that DFS visits w, for any node in C_i and C_j, there is a white-path from w to that node.



  - In this case, due to the white-path theorem, the lemma again holds.

Computing SCC

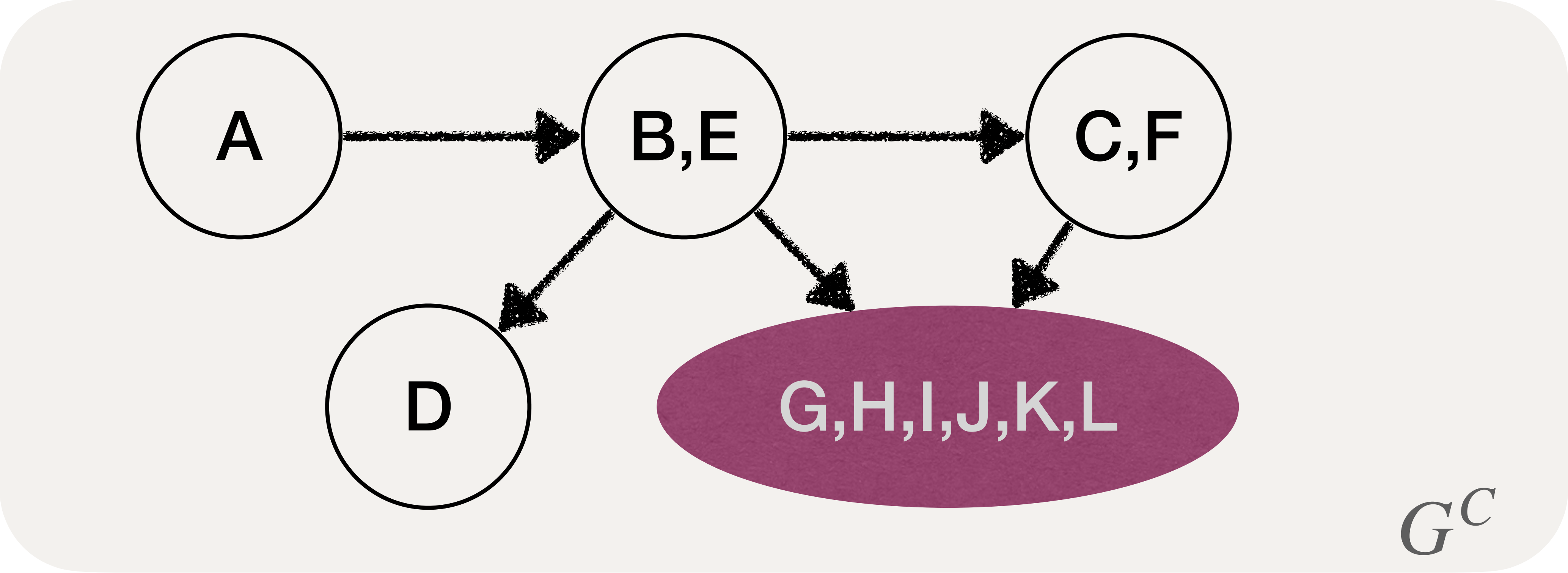
- (1) How to identify a node that is in a sink SCC?
- (2) What to do when the first SCC is done?

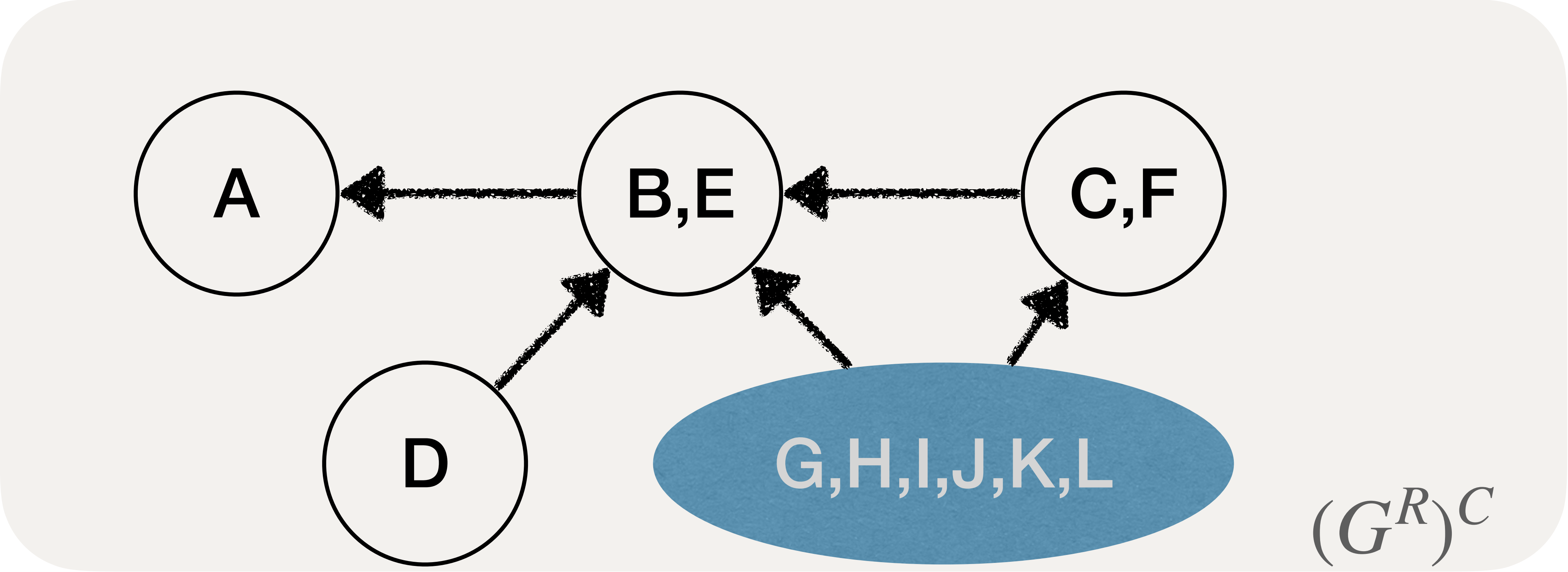
Lemma For any edge $(u, v) \in E(G^R)$, if $u \in C_i$ and $v \in C_j$, then $\max_{u \in C_i}\{u.f\} > \max_{v \in C_j}\{v.f\}$

- Compute G^R in $O(n+m)$ time, do DFS in G^R and find the node with max finish time.
  - This node is in a source SCC of G^R

Computing SCC

- (1) How to identify a node that is in a sink SCC?
- (2) What to do when the first SCC is done?
- For remaining nodes in G, the node with max finish time (in DFS of G^R) is again in a sink SCC, for whatever remains of G.





Computing SCC

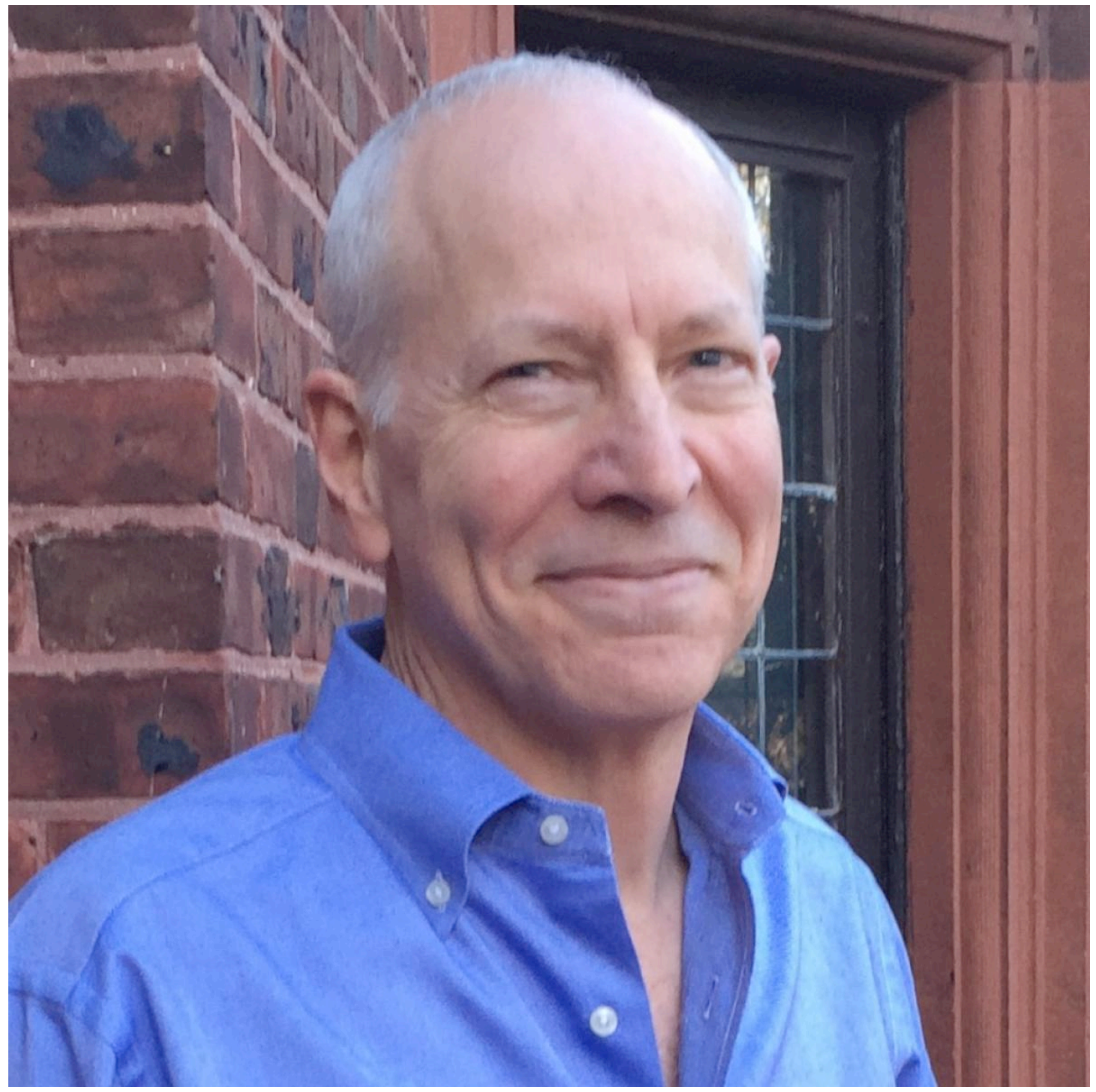
- Algorithm Description:
  - Compute G^R.
  - Run DFS on G^R and record finish times f.
  - Run DFS on G, but in `DFSAll`, process nodes in decreasing order of f.
  - Each DFS tree is a SCC of G.
- Time complexity is $O(n+m)$:
  - $O(n+m)$ time for computing G^R.
  - Two passes of DFS, each costing $O(n+m)$.

Can we be faster (even if just with smaller constant)?

*Tarjan's SCC Algorithm

*Tarjan's SCC Algorithm

- if we start from a node in a sink SCC, then we explore exactly nodes in that SCC and stop!
  - But how to find such a node?
- Previous algorithm's approach:
  - A node in a source SCC in G^R must be in a sink SCC in G.
- Tarjan comes up with a method to identify a node in some sink SCC directly!

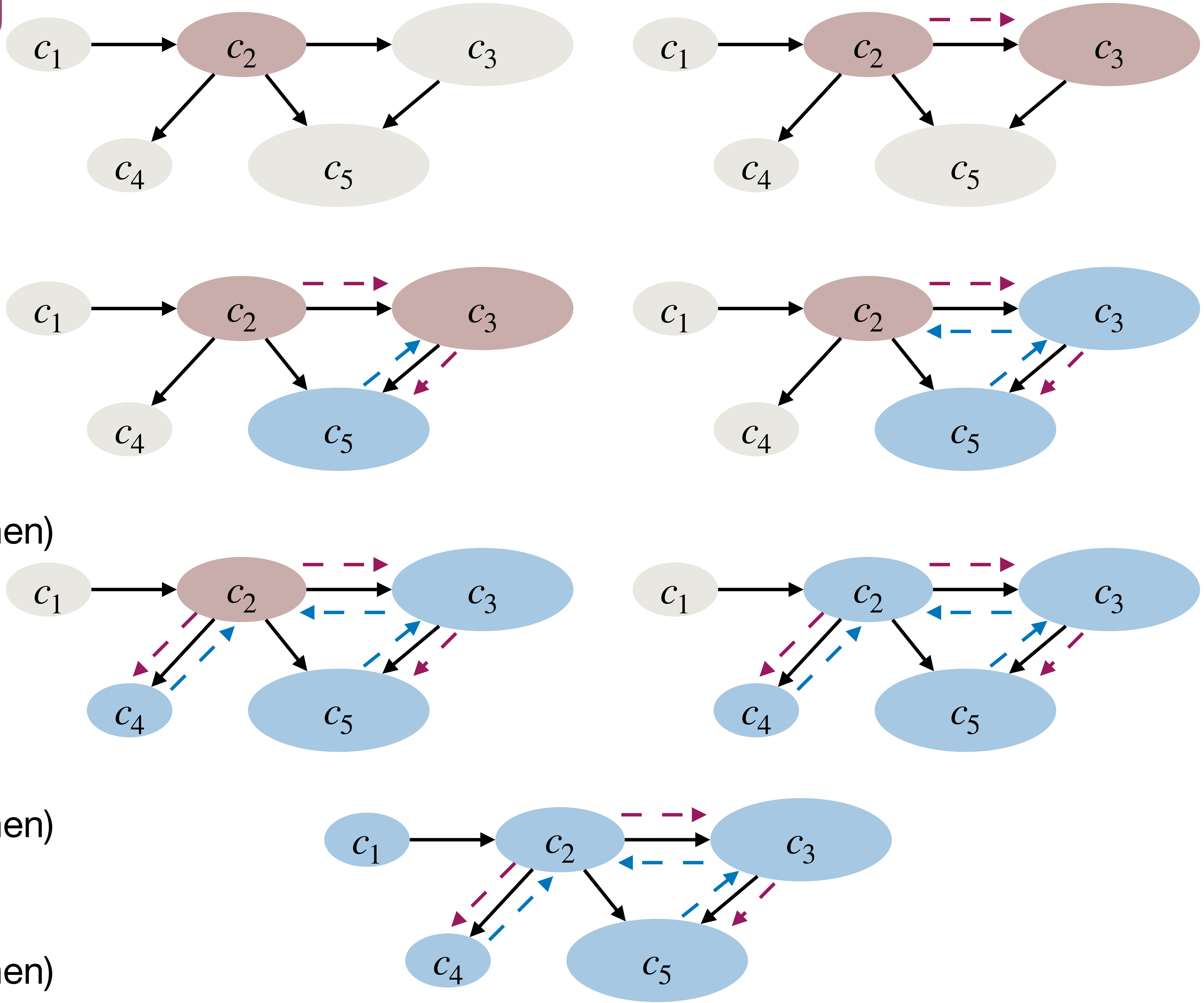


Robert Tarjan

Tarjan's SCC Algorithm

Let's have a closer look at the order that DFS examines nodes

- First node in C_2 (root of C_2)
- Some nodes in C_2
- First node in C_3 (root of C_3)
- Some nodes in C_3
- First nodes in C_5 (root of C_5)
- All other nodes in C_5 (C_5 is a sink SCC)
- All other nodes in C_3 (C_3 becomes a sink SCC by then)
- Some nodes in C_2
- First nodes in C_4 (root of C_4)
- All other nodes in C_4 (C_4 is a sink SCC)
- All other nodes in C_2 (C_2 becomes a sink SCC by then)
- First node in C_1 (root of C_1)
- All other nodes in C_1 (C_1 becomes a sink SCC by then)



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Tarjan's SCC Algorithm

Let's have a closer look at the order that DFS examines nodes

stack bottom

- First node in C_2 (root of C_2)
- Some nodes in C_2
- First node in C_3 (root of C_3)
- Some nodes in C_3
- First nodes in C_5 (root of C_5)
- All other nodes in C_5 (C_5 is a sink SCC)

en)

If we can identify root of C_5, call it r_5, then all nodes visited during DFS starting from r_5 are the nodes in C_5.

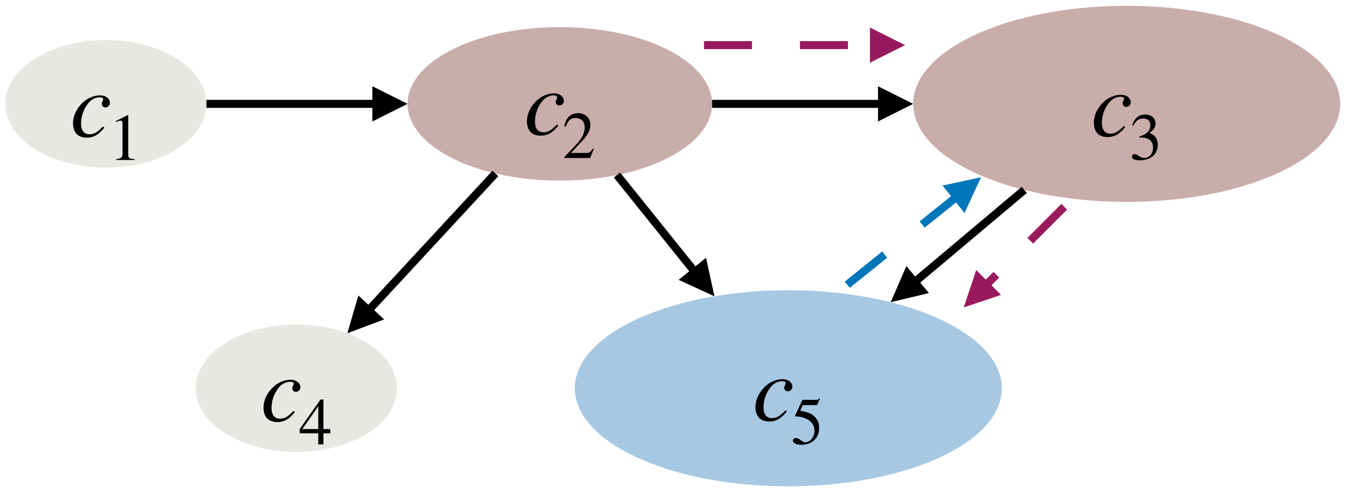
- First nodes in C_4 (root of C_4)

If we push a node to a stack when it is discovered, when DFS returns from r_5, all nodes above r_5 in the stack are in C_5 and can be popped!

en)

- First node in C_1 (root of C_1)
- All other nodes in C_1 (C_1 becomes a sink SCC by then)

stack top



Tarjan's SCC Algorithm

Let's have a closer look at the order that DFS examines nodes

stack bottom

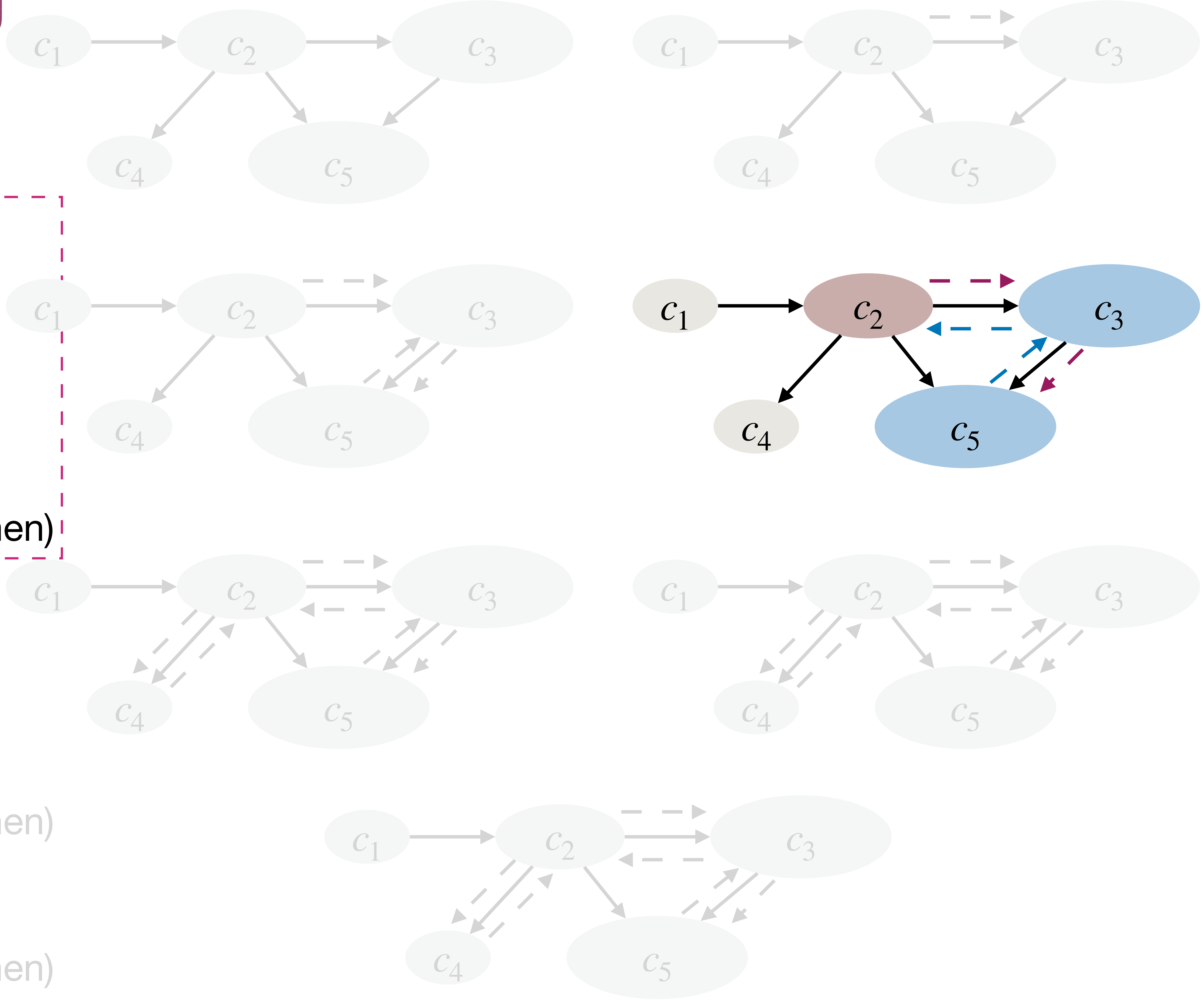
- First node in C_2 (root of C_2)
- Some nodes in C_2
- First node in C_3 (root of C_3)
- Some nodes in C_3
- First nodes in C_5 (root of C_5)
- All other nodes in C_5 (C_5 is a sink SCC)
- All other nodes in C_3 (C_3 becomes a sink SCC by then)

Given that we know nodes in C_5, , if we can identify root of C_3, call it r_3, then all nodes not in C_5 visited during DFS starting from r_3 are the nodes in C_3.

If we push a node to a stack when it is discovered, when DFS returns from r_3, all nodes above r_3 in the stack are in C_3 and can be popped!

- All other nodes in C_1 (C_1 becomes a sink SCC by then)

stack top



Tarjan's SCC Algorithm

Let's have a closer look at the order that DFS examines nodes

stack bottom

- First node in C_2 (root of C_2)
- Some nodes in C_2

If we can identify root of C_4, call it r_4, then all nodes visited during DFS starting from r_4 are the nodes in C_4.

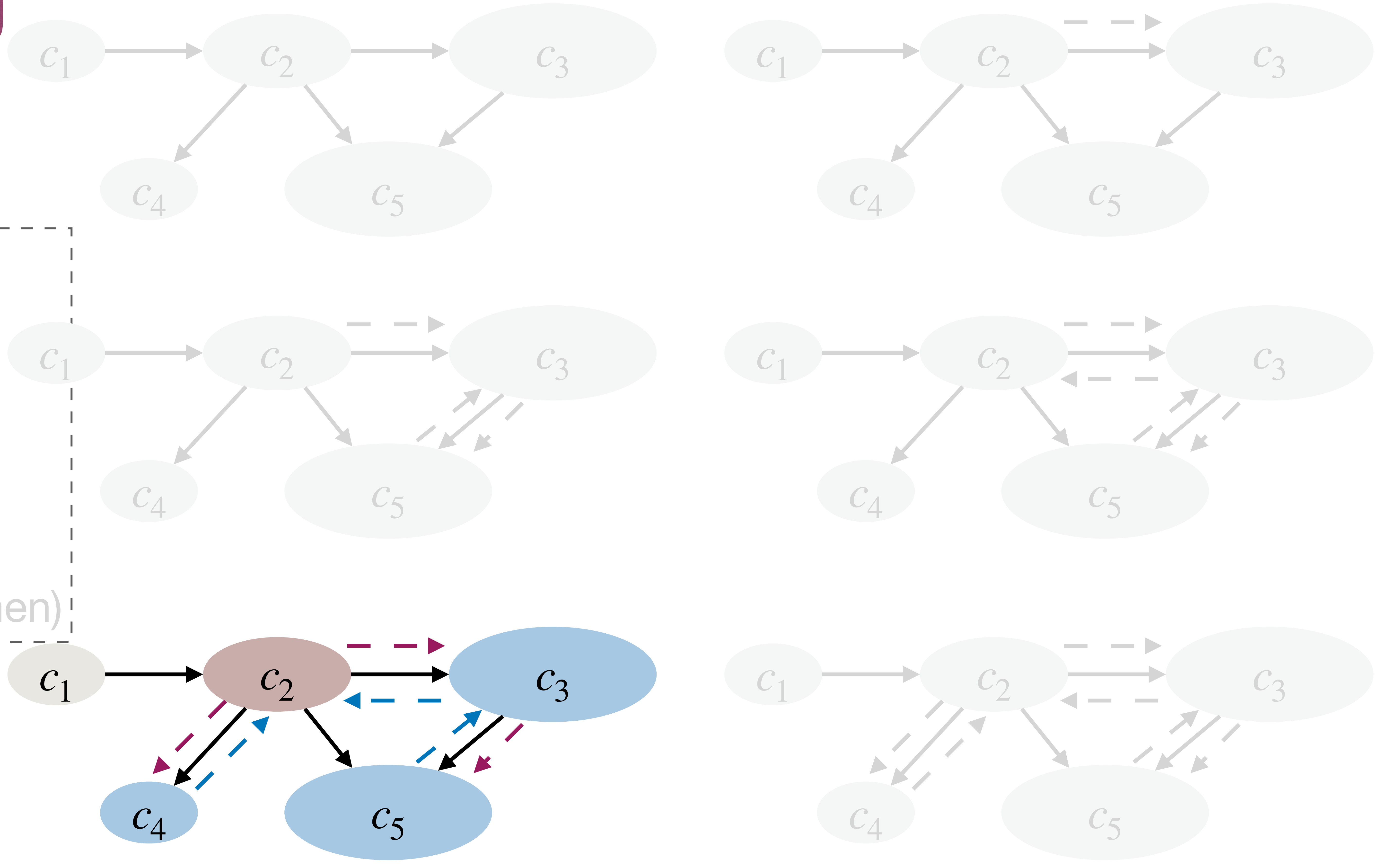
- First nodes in C_4 (root of C_4)

If we push a node to a stack when it is discovered, when DFS returns from r_4, all nodes above r_4 in the stack are in C_4 and can be popped!

en)

- Some nodes in C_2
- First nodes in C_4 (root of C_4)
- All other nodes in C_4 (C_4 is a sink SCC)
- All other nodes in C_2 (C_2 becomes a sink SCC by then)
- First node in C_1 (root of C_1)
- All other nodes in C_1 (C_1 becomes a sink SCC by then)

stack top



Tarjan’s SCC Algorithm

Let’s have a closer look at the order that DFS examines nodes

stack bottom

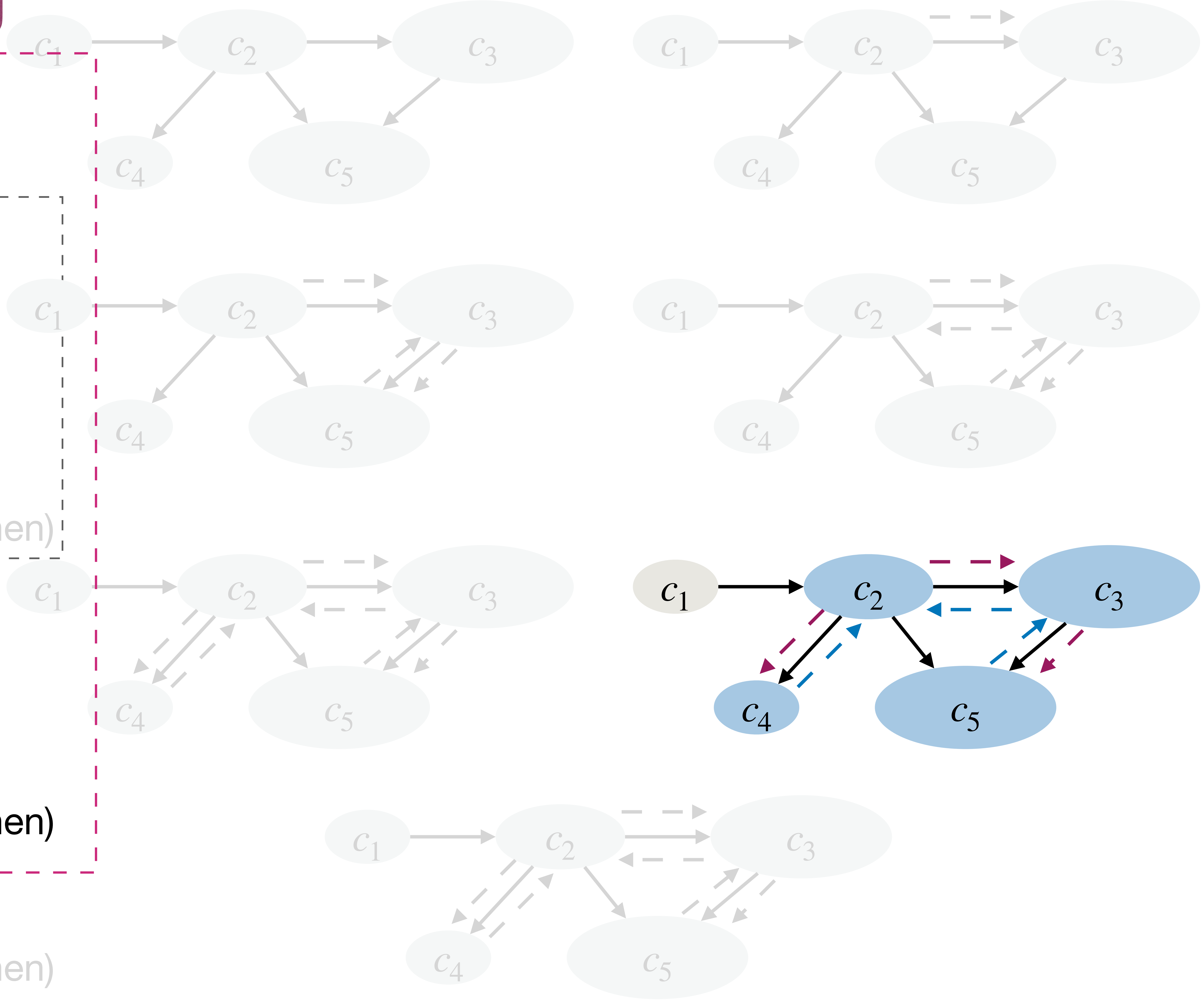
- First node in C_2 (root of C_2)
- Some nodes in C_2

Given that we know nodes in C_5 & C_4 & C_3, if we can identify root of C_2, call it r_2, then all nodes not in C_5 & C_4 & C_3 visited during DFS starting from r_2 are the nodes in C_2.

If we push a node to a stack when it is discovered, when DFS returns from r_2, all nodes above r_2 in the stack are in C_2 and can be popped!

- Some nodes in C_2
  - First nodes in C_4 (root of C_4)
  - All other nodes in C_4 (C_4 is a sink SCC)
- All other nodes in C_2 (C_2 becomes a sink SCC by then)
- First node in C_1 (root of C_1)
- All other nodes in C_1 (C_1 becomes a sink SCC by then)

stack top



Tarjan's SCC Algorithm

Let's have a closer look at the order that DFS examines nodes

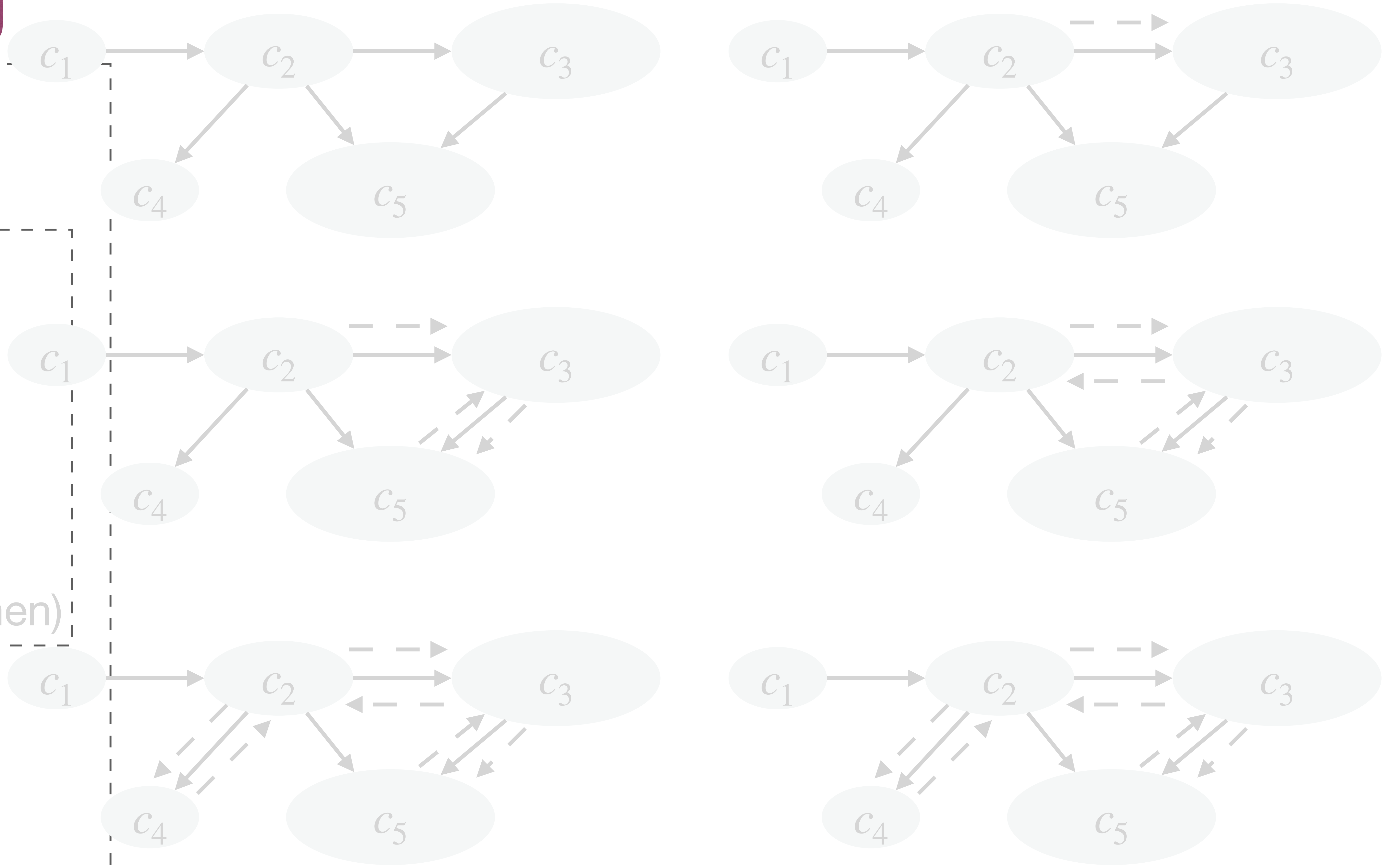
stack bottom

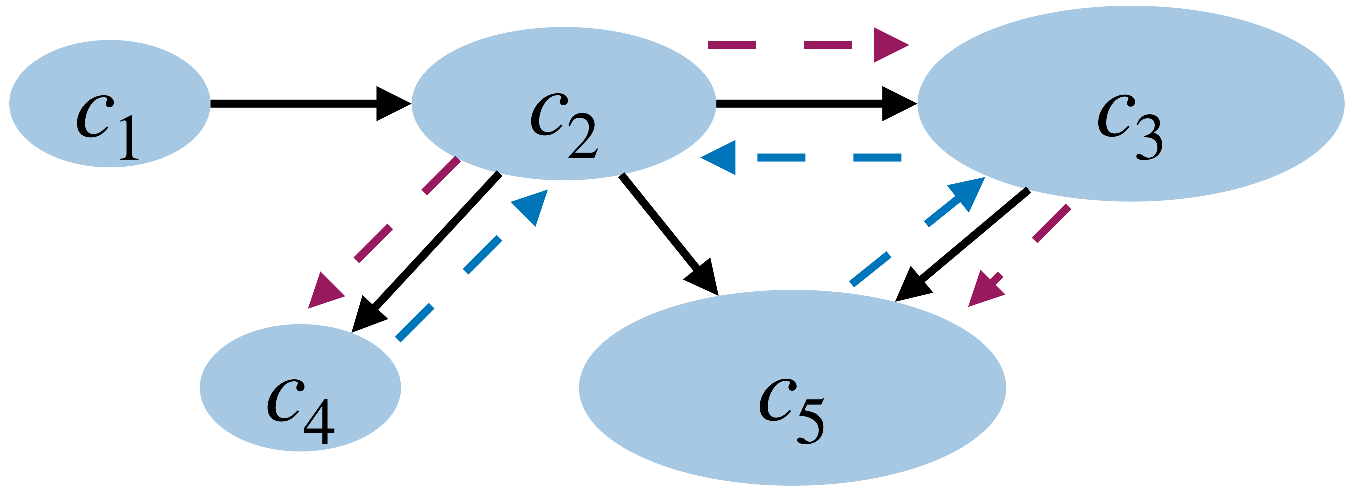
- First node in C_2 (root of C_2)
- Some nodes in C_2
- First node in C_3 (root of C_3)
- Some nodes in C_3
- First nodes in C_5 (root of C_5)
- All other nodes in C_2 (C_2 becomes a sink SCC by then)
- First node in C_1 (root of C_1)
- All other nodes in C_1 (C_1 becomes a sink SCC by then)

stack top

Given that we know nodes in C_2, if we can identify root of C_1, call it r_1, then all nodes not in C_1 visited during DFS starting from r_1 are the nodes in C_1.

If we push a node to a stack when it is discovered, when DFS returns from r_1, all nodes above r_1 in the stack are in C_1 and can be popped!





Tarjan's SCC Algorithm

Let's have a closer look at the order that DFS examines nodes

stack bottom

- First node in C_2 (root of C_2)
- Some nodes in C_2
- First node in C_3 (root of C_3)

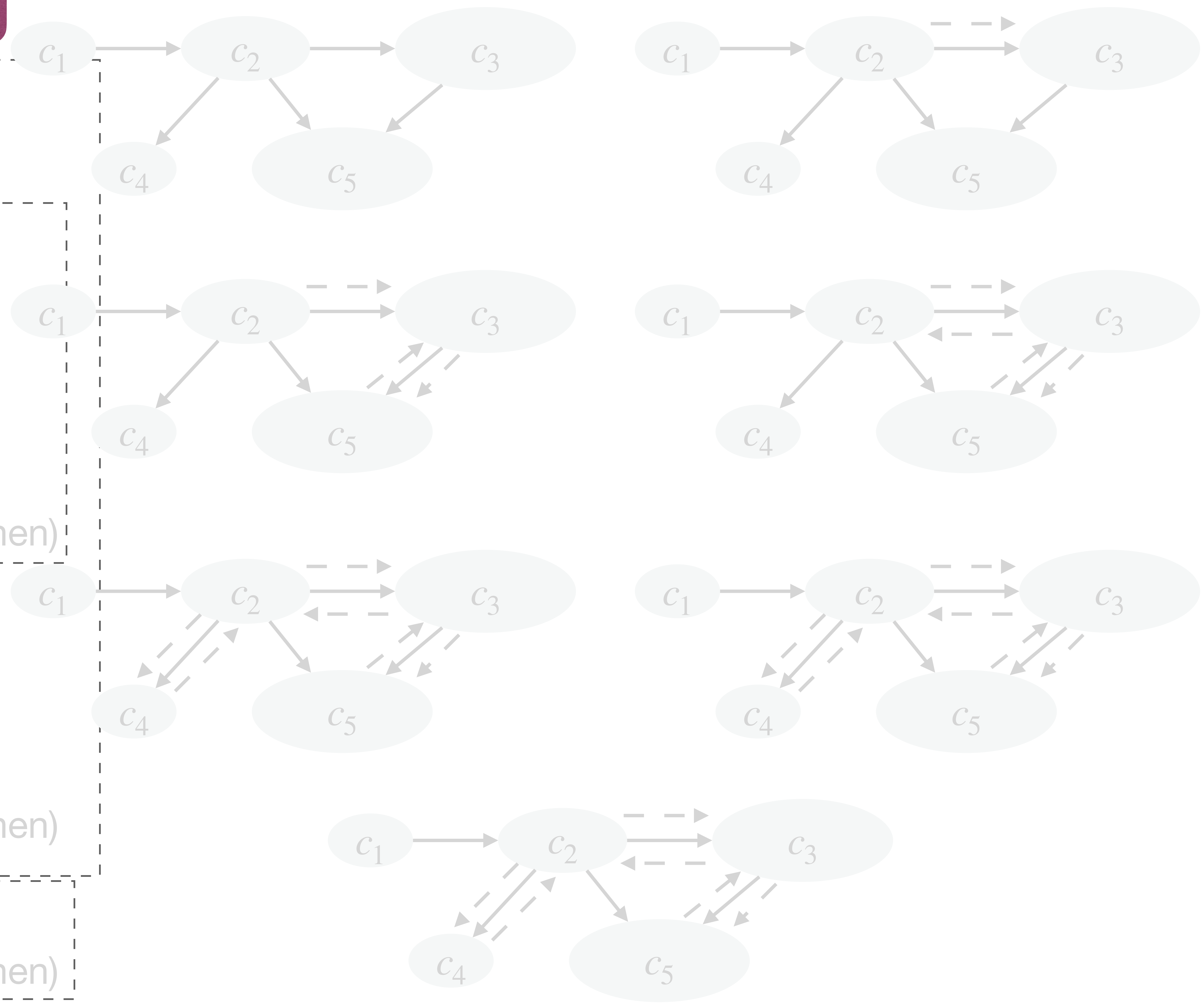
For each SCC C_i, let r_i be its root. If we push a node to a stack when it is discovered, when DFS returns from r_i, all nodes above r_i in the stack are in C_i and can be popped!

But how to identify each root r_1?

...en)

- Some nodes in C_2
- First nodes in C_4 (root of C_4)
- All other nodes in C_4 (C_4 is a sink SCC)
- All other nodes in C_2 (C_2 becomes a sink SCC by then)
- First node in C_1 (root of C_1)
- All other nodes in C_1 (C_1 becomes a sink SCC by then)

stack top



Tarjan's method to identify root of SCC

- Fix some DFS process, for each vertex v, let C_v be the SCC that v is in. Then, $low(v)$ is the smallest discovery time among all nodes in C_v that are reachable from v via a path of tree edges followed by at most one non-tree edge.

- By definition, $low(v) \leq v.d$ as v is reachable from itself.

Lemma Node v is the root of a SCC iff $low(v) = v.d$

Tarjan's method to identify root of SCC

Lemma Node v is the root of a SCC iff $low(v) = v.d$

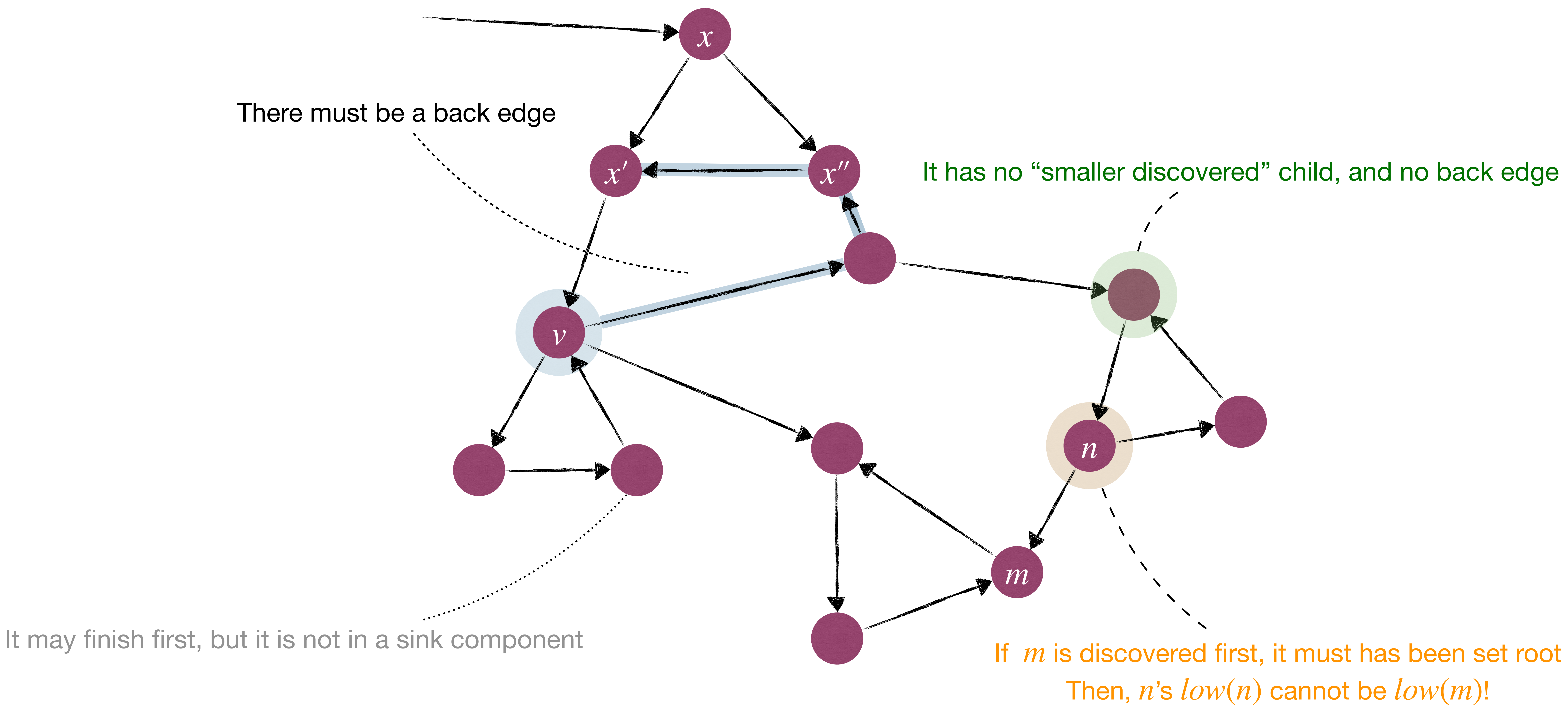
- Proof of [$\Longrightarrow$] (easy direction)
  - If v is the root of C_v, then it is the first discovered node in C_v.
  - Hence v has the smallest discovery time among all nodes in C_v.
  - By the definition of $low(v)$, clearly $low(v) = v.d$.

Tarjan's method to identify root of SCC

Lemma Node v is the root of a SCC iff $low(v) = v.d$

- Proof of [$\Longleftarrow$] (hard direction)
  - For the sake of contradiction assume $x \neq v$ is the root of C_v. (That is, x is the first discovered node in C_v.)
  - Let $x' \neq v$ be v's parent in the DFS tree. Since C_v is a SCC, v can reach all nodes in C_v, including the ones on path $x \to x'$. Thus, when executing DFS from v, it will examine a path containing zero or more tree edges and then a back edge pointing to some node x'' in path $x \to x'$.
  - But this means $low(v) < v.d$ since $low(v) \leq x''.d < v.d$. Contradiction!

Tarjan’s method to identify root of SCC



Tarjan's SCC Algorithm

- Now we have:
  - For each SCC C_i, let r_i be its root. If we push a node to a stack when it is discovered, when DFS returns from r_i, all nodes above r_i in the stack are in C_i.
  - Let $low(v)$ be the smallest discovery time among all nodes in C_i that are reachable from v via a path of tree edges followed by at most one non-tree edge.
  - Lemma: Node v is the root of a SCC iff $low(v) = v.d$

Tarjan's SCC Algorithm

Tarjan(G):

$time := 0$

Stack S

for **each** v **in** V

$v.root := NIL$

$v.visited := False$

for **each** v **in** V

if $!v.visited$

$TarjanDFS(v)$

TarjanDFS(v):

$v.visited := True, time := time + 1$

$v.d := time, v.low := v.d$

S.**push**(v)

for **each** $edge(v, w)$

if $!w.visited$ // tree edge

$TarjanDFS(w)$

$v.low :=$ **min**$(v.low, w.low)$

else if $w.root = NIL$ // non tree edge in C_v

$v.low :=$ **min**$(v.low, w.d)$

if $v.low = v.d$

repeat

$w := S$.**pop**$(), w.root := v$

until $w = v$

Time complexity is $O(m + n)$
(One DFS pass, and push/pop once for each node)

Further reading

- [CLRS] Ch.22
- [Erickson] Ch.6