智能软件与工程学院
School of Intelligent Software and Engineering

单源最短路径
Single-Source Shortest Path

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The slides are mainly adapted from the original ones shared by Chaodong Zheng and Kevin Wayne. Thanks for their supports!

The Shortest Path Problem

- Given a map, what's the **shortest path** from s to t?

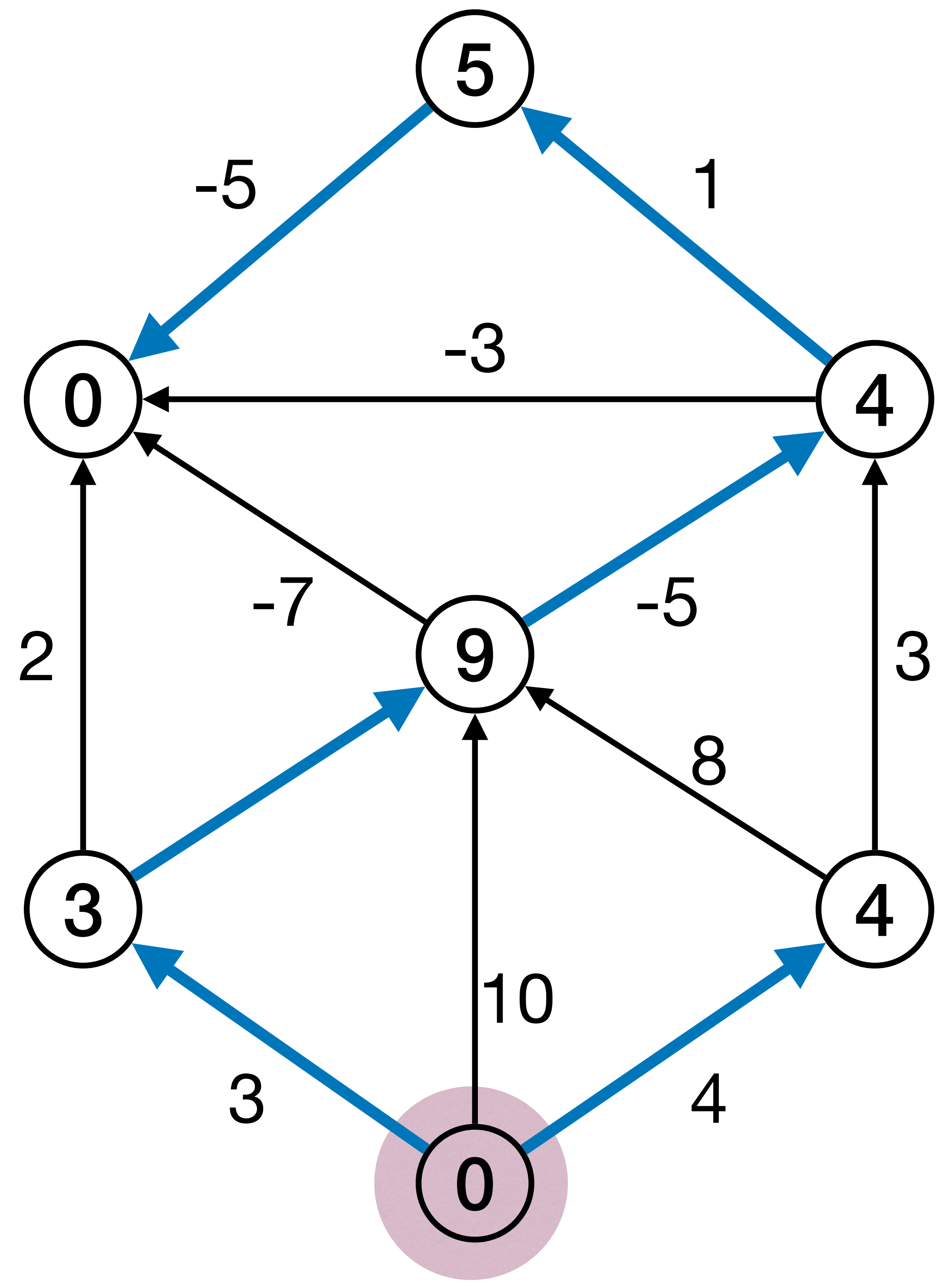
- Consider a graph $G = (V, E)$ and a weight function $w$that associates a real-valued weight $w(u, v)$ to each edge (u, v). Given s and t in V, what's the **min weight path** from s to t?

The Shortest Path Problem

- **Weights** are not always lengths.
    - E.g., time**,** cost**, …** to walk the edge.
- The graph can be **directed**.
    - Thus $w(u, v) \neq w(v, u)$ possible.
- **Negative** edge weight **allowed**.
- **Negative** cycle ***not*** allowed.
    - Problem not well-defined then.

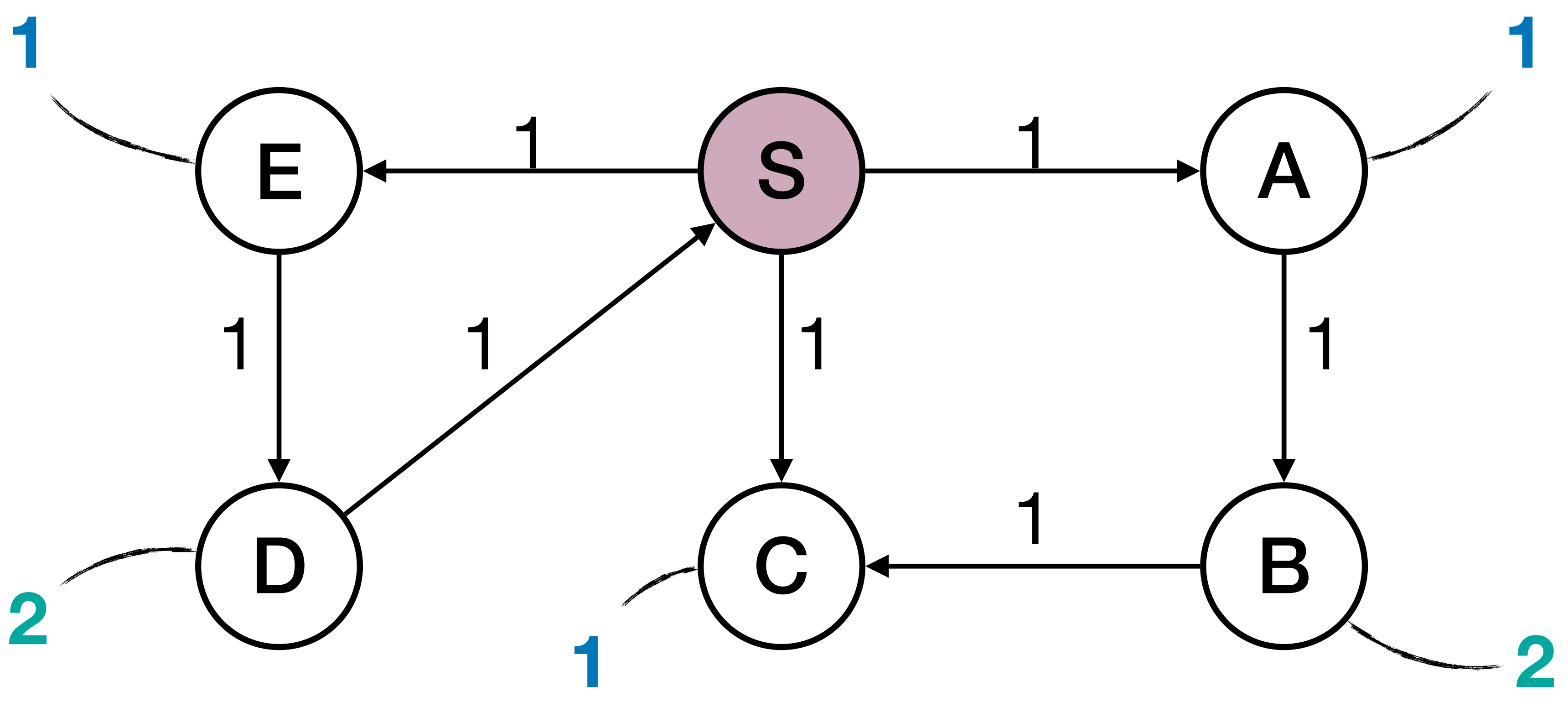
Single-Source Shortest Path (SSSP)

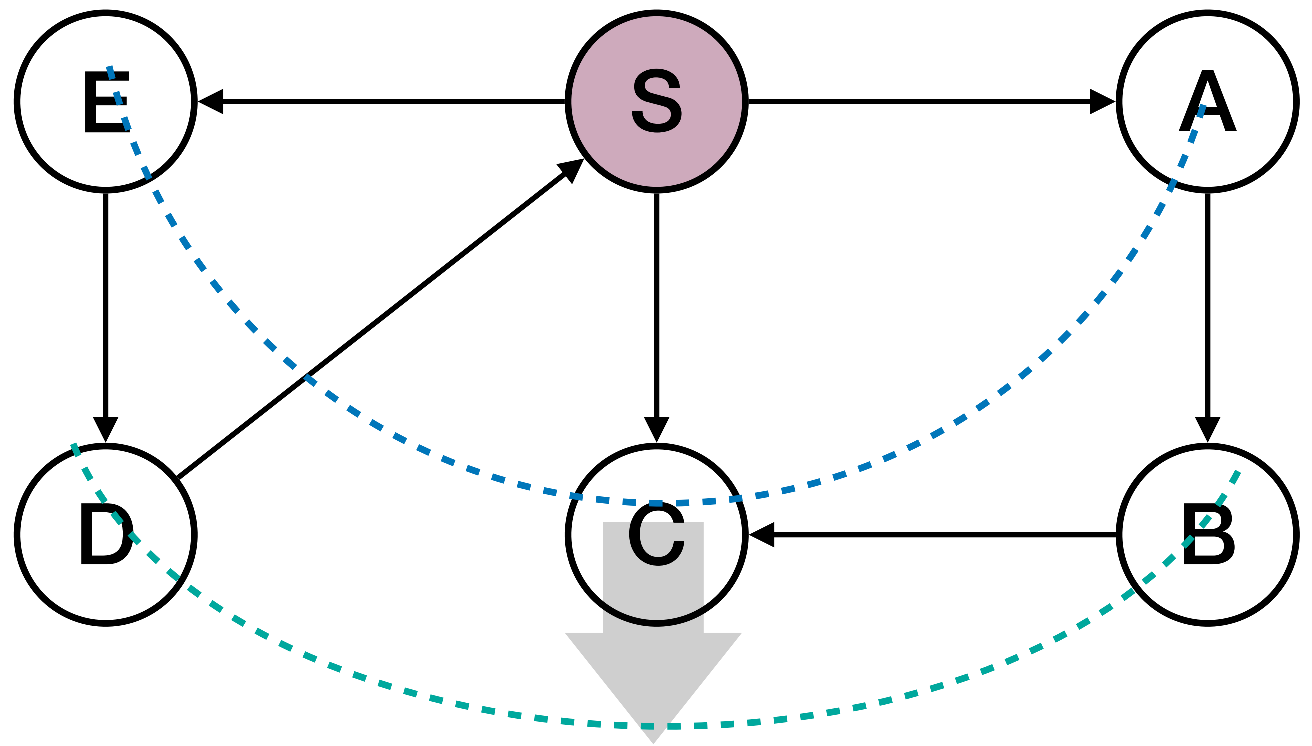
- **The SSSP Problem:** Given a graph $G = (V, E)$ and a weight function w, given a source node s, find a shortest path from s to every node $u \in V$.
- Consider *directed* graphs *without* negative cycle.
  - **Case 1**: Unit weight.
  - **Case 2**: Arbitrary positive weight.
  - **Case 3**: Arbitrary weight without cycle.
  - **Case 4**: Arbitrary weight.



SSSP in unit weight graphs

- How to solve SSSP in an unit weight graph?
  - That is, a graph in which each edge is of weight **1**.
- “Traverse by layer” in an unweighted graph!
  - Visit all distance d nods before visiting any distance $d + 1$ node.
  - Simple, just use BFS!





SSSP in positive weight graphs

- Solve SSSP in a graph with *arbitrary positive weights*?
- Extension of unit graph SSSP algorithm:
  - Add dummy nodes on edges so graph becomes unit weight graph.
  - Run BFS on the resulting graph.

The problem is that it is too slow when edge weights are large!

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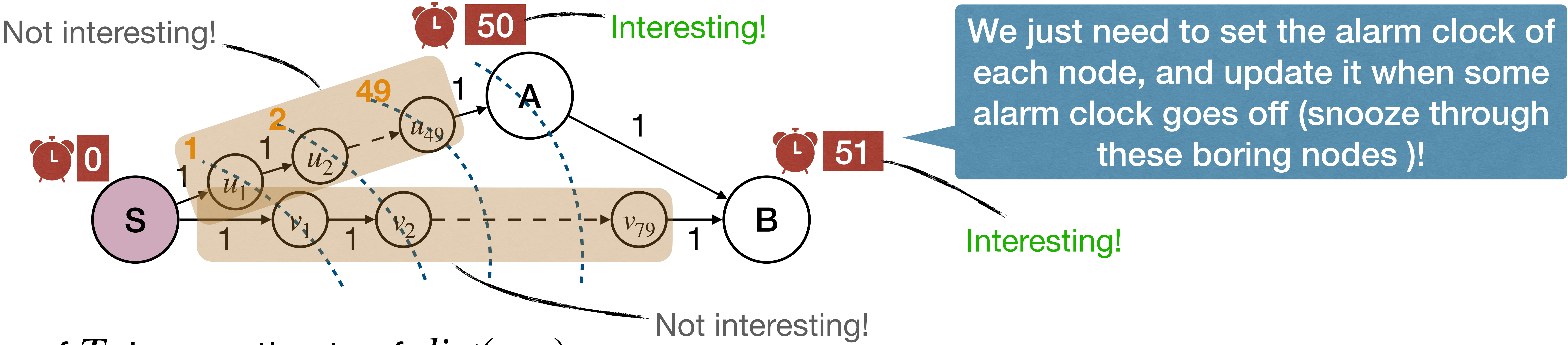
Extension of the BFS algorithm

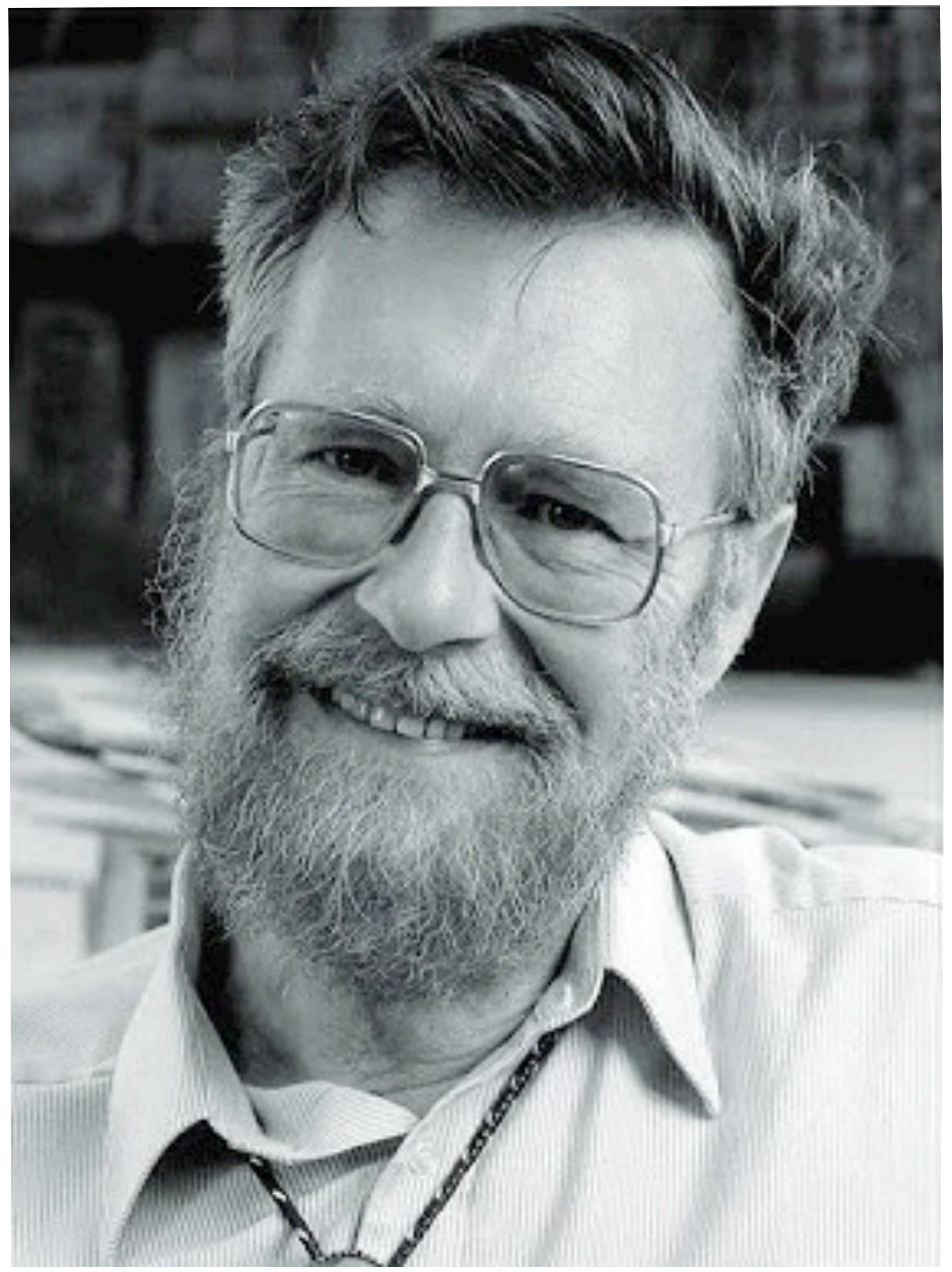
- To save time, bypass the events that process dummy nodes!
  - Imagine we have an alarm clock T_u for each node u.
  - Alarm for source node s goes off at time 0.
  - If T_u goes off, for each edge (u, v), update $T_v = \min\{T_v, T_u + w(u, v)\}$



- At any time, value of T_u is an estimate of $dist(s, u)$.
- At any time, $T_u \geq dist(s, u)$, with equality holds when T_u goes off.

Dijkstra's algorithm

- How to implement the “alarm clock”?
  - Use priority queue (such as binary heap).



Edsger W. Dijkstra

DijkstraSSSP(G, s):

Shortest-path Tree
(Similar to BFS tree.)

for each u **in** V
    $u.dist := INF$, $u.parent := NIL$
$s.dist := 0$
$Build\ priority\ queue\ Q\ based\ on\ dist$
while $!Q.empty()$
    $u := Q.ExtractMin()$
    for each edge (u,v) **in** E
        if $v.dist > u.dist + w(u, v)$
            $v.dist := u.dist + w(u, v)$
            $v.parent := u$
            $Q.UpdateKey(v)$

Dijkstra's algorithm

- Correctness of Dijkstra's algorithm?
  - Similar to the correctness proof of BFS.
- Efficiency of Dijkstra's algorithm?
  - $O((n+m)\cdot\log n)$ when using a binary heap.

DijkstraSSSP(G, s):

for **each** u **in** V — $O(n)$

    $u.dist := INF, u.parent := NIL$

$s.dist := 0$

Build priority queue Q based on dist — $O(n)$

while $!Q.empty()$

    $u := Q.ExtractMin()$ — $O(n\log n)$

    for **each** edge (u,v) **in** E — $O(m\log n)$

        if $v.dist > u.dist + w(u,v)$

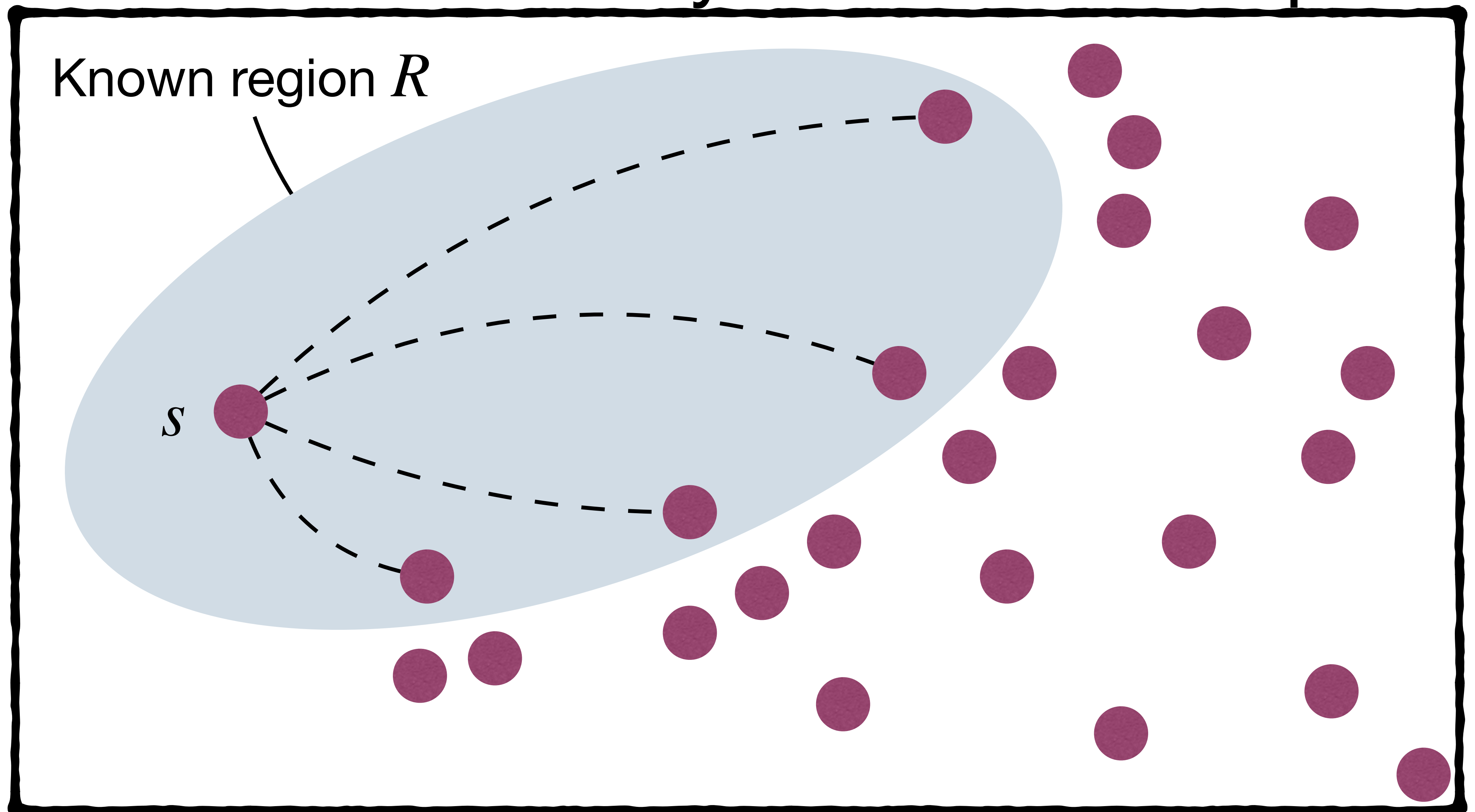
            $v.dist := u.dist + w(u,v)$

            $v.parent := u$

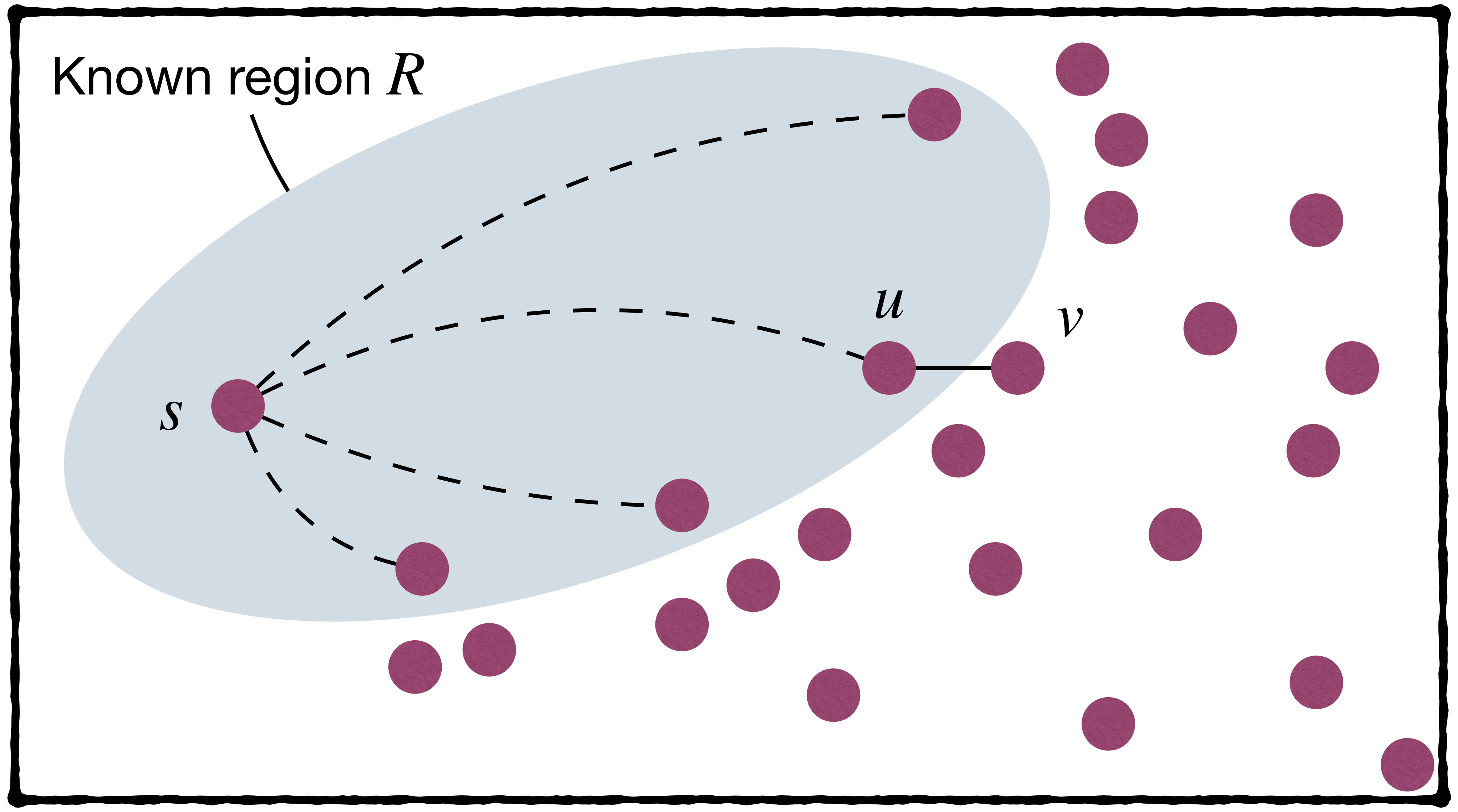
            $Q.UpdateKey(v)$

Alternative derivation of Dijkstra's algorithm

- What's BFS doing: ***expand*** outward from s, growing the ***region*** to which distances and shortest paths are known.
  - Growth should be ***orderly***: closest nodes first.
- Given "known region R",
  - how to identify the node to expand to?

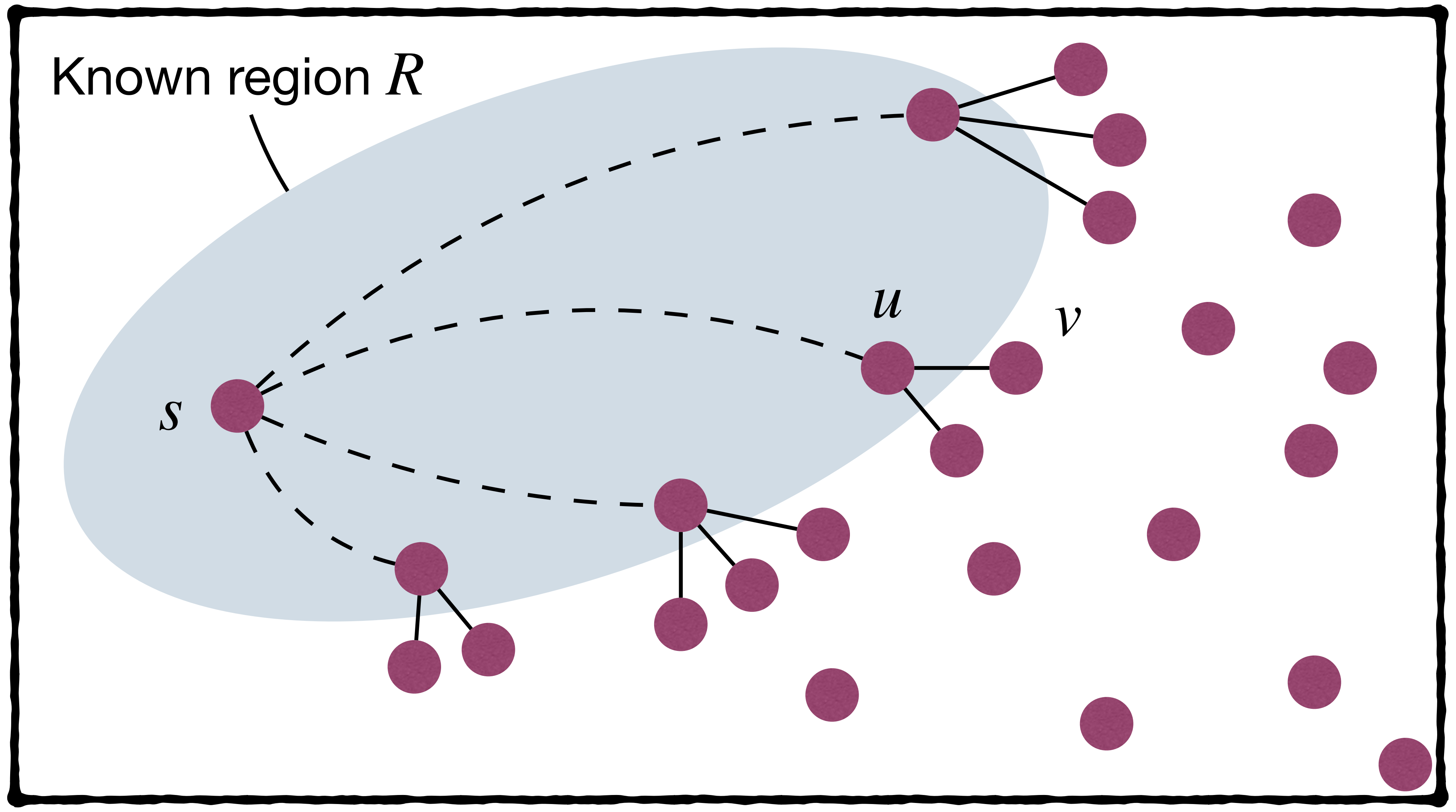


Alternative derivation of Dijkstra's algorithm



- Given "known region R", assume v is such node to expand to (that is, the next closet node to s), let the shortest path from s to v is $s \rightsquigarrow v$.
  - It must be $dist(s, v) \geq dist(s, v')$, for any $v' \in R$. (Otherwise it is already $v \in R$)
  - Let the last node of the path $s \rightsquigarrow v$ before v be u, then it must be $u \in R$. (Otherwise v is not the next closet node to s)

Alternative derivation of Dijkstra's algorithm



- Given "known region R",
  - Find $\min_{u' \in R, v' \in V-R} \{dist(s, u') + w(u', v')\}$,
  - Any satisfied node v is the next node to expand to (the next closet node to s)

Alternative derivation of Dijkstra's algorithm

- BFS expands outward from s, growing the region to which distances and shortest paths are known.
  - Given "known region R", expend to the node with $\min_{u' \in R, v' \in V-R} \{dist(s, u') + w(u', v')\}$.

DijkstraSSSPAbs(G, s):

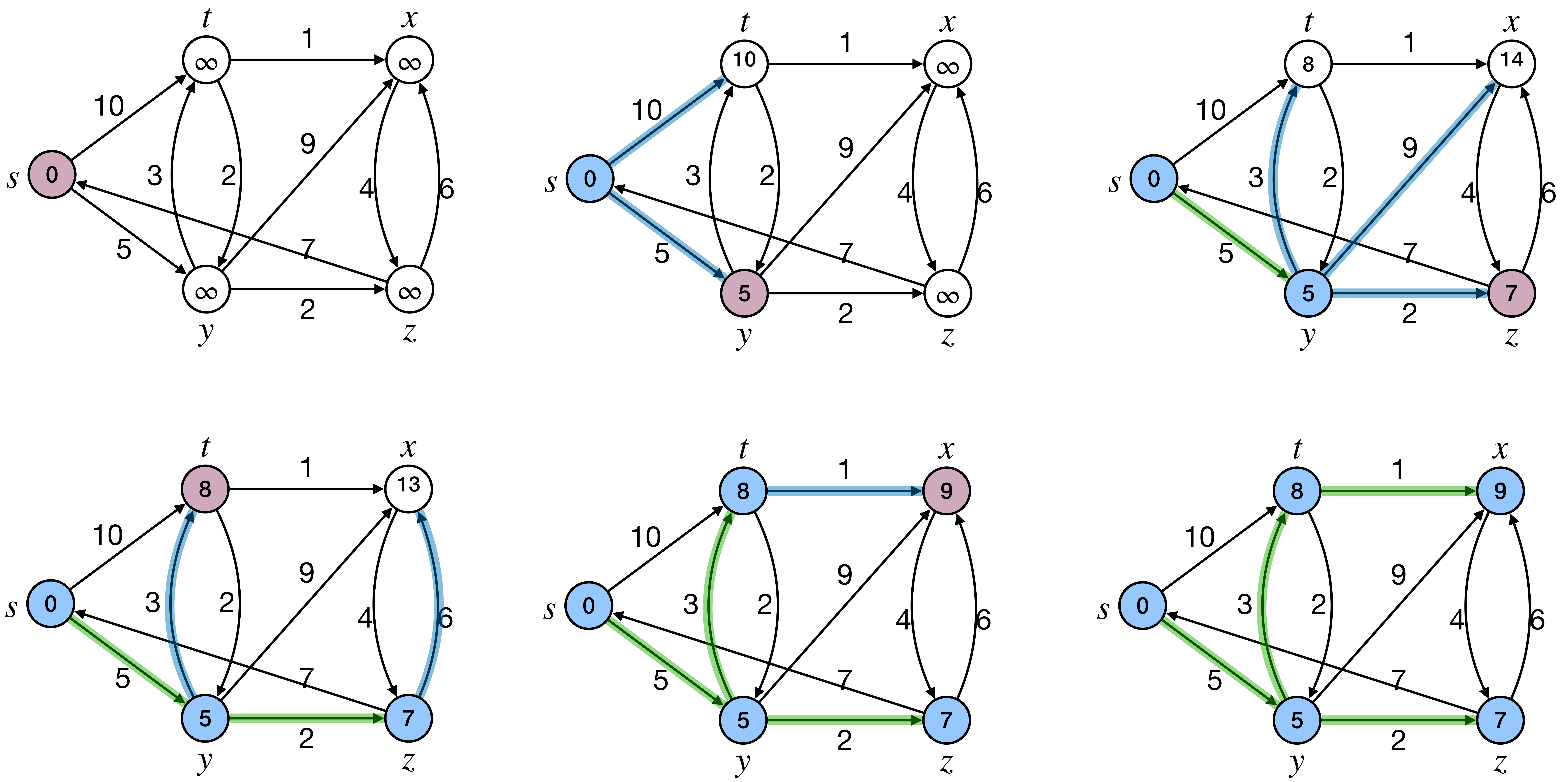
for each u **in** V
 $u.dist := INF$
$s.dist := 0$
$R := \emptyset$
while $R \,!= V$
 Find node v in V - R with min v.dist
 Add v to R
 for each edge (v, z) in E
  if $z.dist > v.dist + w(v, z)$
   $z.dist := v.dist + w(v, z)$

Priority queue implementation

DijkstraSSSP(G, s):

for each u **in** V
 $u.dist := INF, u.parent := NIL$
$s.dist := 0$
Build priority queue Q based on dist
while $!Q.empty()$
 $u := Q.ExtractMin()$
 for each edge (u,v) **in** E
  if $v.dist > u.dist + w(u, v)$
   $v.dist := u.dist + w(u, v)$
   $v.parent := u$
   $Q.UpdateKey(v)$

Alternative derivation of Dijkstra's algorithm



DFS, BFS, Prim, Dijkstra, and others…

```
DFSIterSkeleton(G, s):
Stack Q
Q.push(s)
while !Q.empty()
    u := Q.pop()
    if  !u.visited
        u.visited := True
        for each edge (u, v) in E
            Q.push(v)
```

```
BFSSkeletonAlt(G, s):
FIFOQueue Q
Q.enque(s)
while !Q.empty()
    u := Q.dequeue()
    if  !u.visited
        u.visited := True
        for each edge (u, v) in E
            Q.enque(v)
```

```
PrimMSTSkeleton(G, x):
PriorityQueue Q
Q.add(x)
while !Q.empty()
    u := Q.remove()
    if  !u.visited
        u.visited := True
        for each edge (u, v) in E
            if !v.visited and …
                Q.update(v, …)
```

```
DijkstraSSSPSkeleton(G, x):
PriorityQueue Q
Q.add(x)
while !Q.empty()
    u := Q.remove()
    if  !u.visited
        u.visited := True
        for each edge (u, v) in E
            if !v.visited and …
                Q.update(v, …)
```

```
GraphExploreSkeleton(G, s):
GenericQueue Q
Q.add(s)
while !Q.empty()
    u := Q.remove()
    if  !u.visited
        u.visited := True
        for each edge (u, v) in E
            Q.add(v)
```

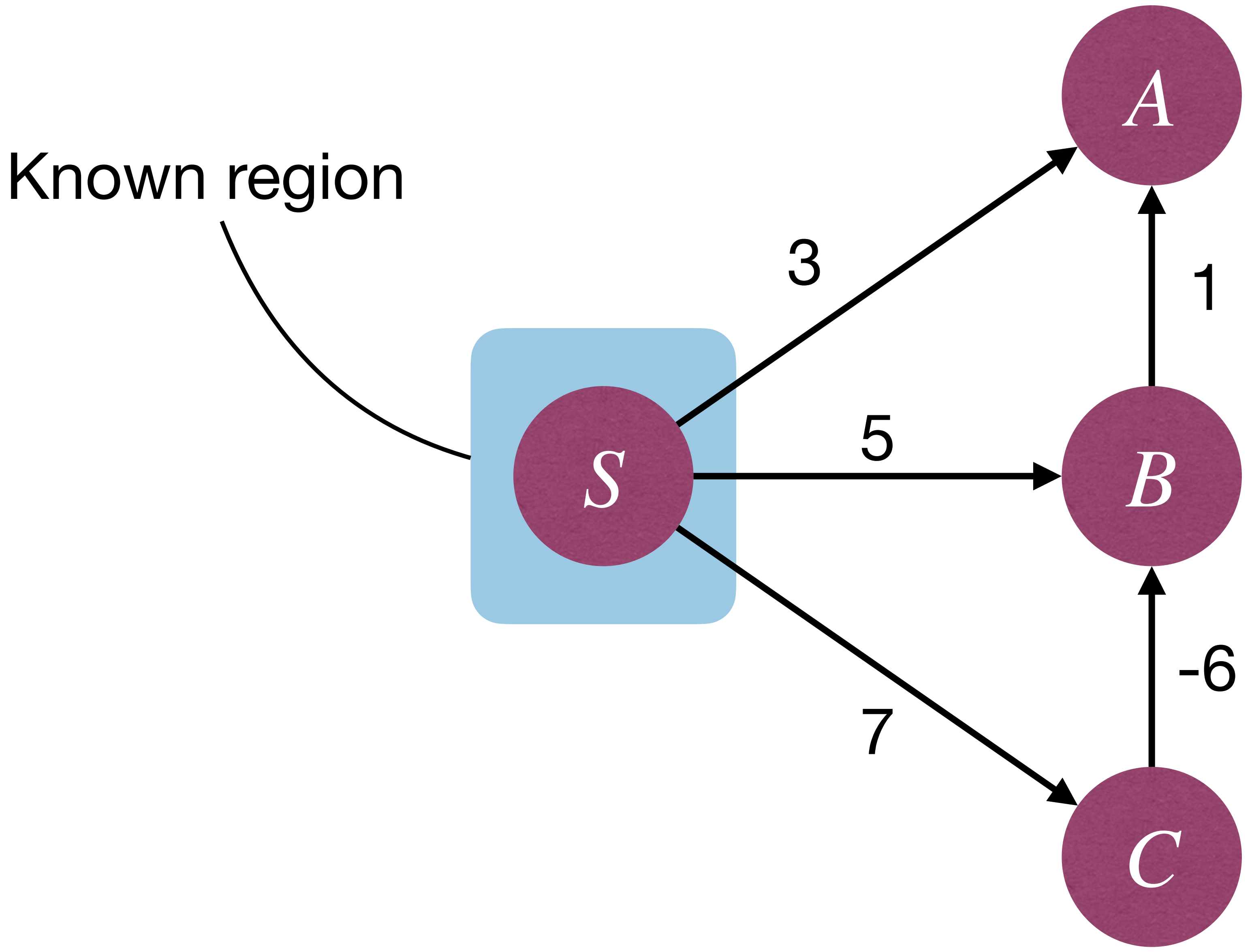
SSSP in graphs with negative weights

- Dijkstra's algorithm no longer works!
- Why would this happen?
- Dijkstra's algorithm for finding next closest node to expend to:
- Given "known region R", find $\min_{u' \in R, v' \in V-R} \{dist(s,u') + w(u',v')\}$.

  - This is because: Let the last node of the path $s \rightsquigarrow v$ before v be u, then it must be $u \in R$. (Otherwise v is not the next closet node to s)

However, negative edge makes this does not hold!

SSSP in graphs with negative weights



Shortest distance from S to node A is 3? No!!!
Try $S \to C \to B \to A$

- "*Shortest path from s to any node v must pass through nodes that are closer than v*" no longer holds!

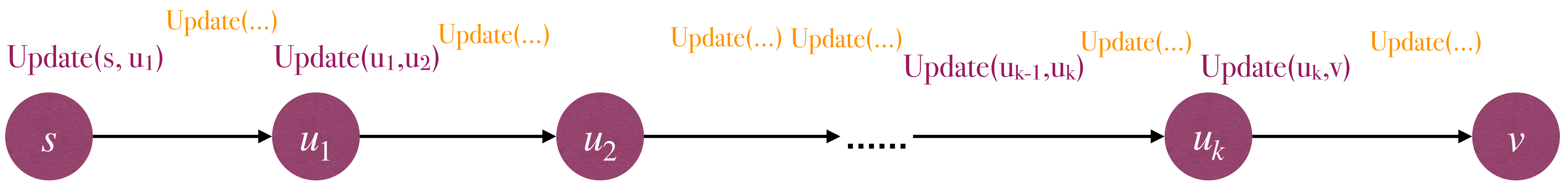
SSSP in graphs with negative weights

- But how $dist$ values are maintained in Dijkstra is helpful:
  - Initially set $s.dist = 0$, and for each node $u \neq s$, set $u.dist = \infty$.
  - When processing edge (u, v), execute procedure `Update(u, v)`: $v.dist = \min\{v.dist, u.dist + w(u, v)\}$
- This way two properties are maintained:
  - For any v, at any time, $v.dist$ is either an **overestimate**, or **correct**.
  - Assume u is the last node on a shortest path from s to v. If $u.dist$ is correct and we run `Update(u, v)`, then $v.dist$ becomes correct.

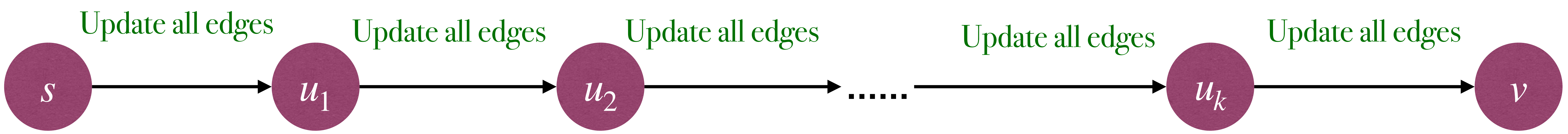
SSSP in graphs with negative weights

- `Update(u,v)` is safe and helpful!
    - [Safe] Regardless of the sequence of `Update` operations we execute, for any node v, value $v.dist$ is either an overestimate or correct.
    - [Helpful] With correct sequence of `Update`, we get correct $v.dist$.

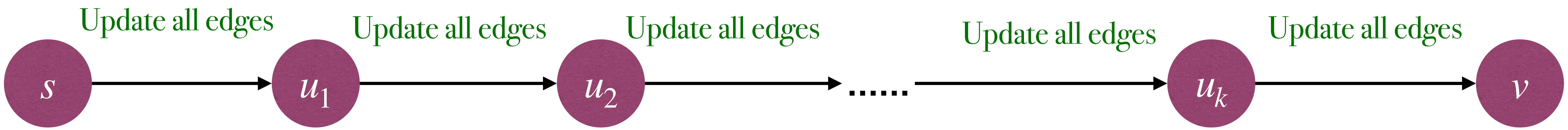
SSSP in graphs with negative weights



- Consider a shortest path from s to v.
  - **Observation 1**: if `Update(s, u₁)`, `Update(u₁,u₂)`, …, `Update(uₖ₋₁,uₖ)`, `Update(uₖ,v)` are executed, then we correctly obtain the shortest path.
  - **Observation 2**: in above sequence, before and after each `Update`, we can add arbitrary `Update` sequence, and still get shortest path from s to v.
- Algorithm: simply `Update` all edges, for $k+1$ times!



SSSP in graphs with negative weights



- But how large is $k + 1$?
  - **Observation 3**: any shortest path cannot contain a cycle. (WHY?)
- Algorithm: simply `Update` <u>all</u> edges, for $n - 1$ times!
  - The Bellman-Ford Algorithm!

The Bellman-Ford Algorithm

- Bellman-Ford Algorithm:
  - `Update` all edges;
  - Repeat above step for $n-1$ times.
- The complexity is : $\Theta\left(n(m+n)\right)$

Richard E. Bellman

Lester Randolph Ford Jr.

<u>BellmanFordSSSP(G, s):</u>

for each u **in** V

  $u.dist := INF, u.parent := NIL$

$s.dist := 0$

repeat n - 1 times

  for each edge (u, v) **in** E

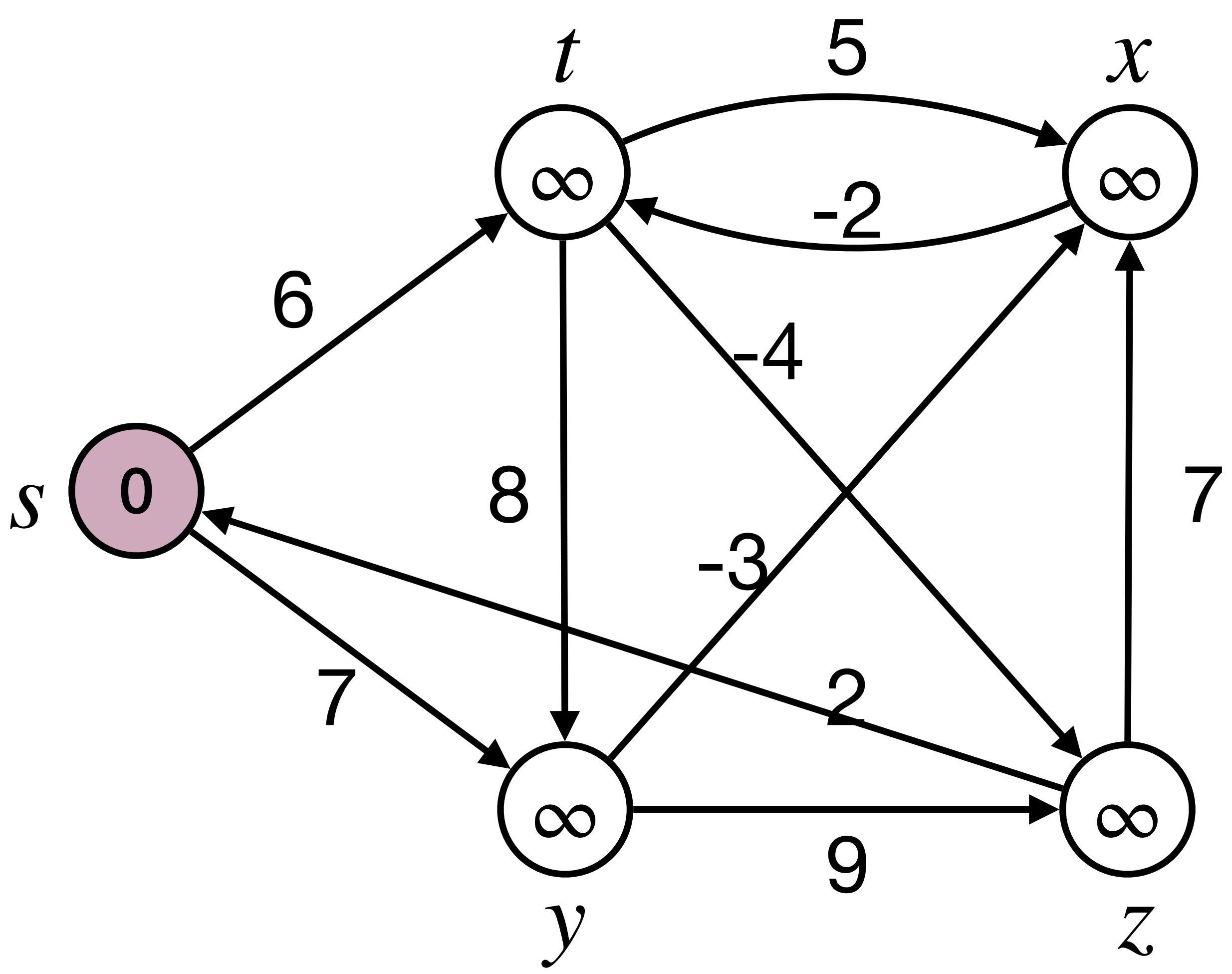
    if $v.dist > u.dist + w(u, v)$

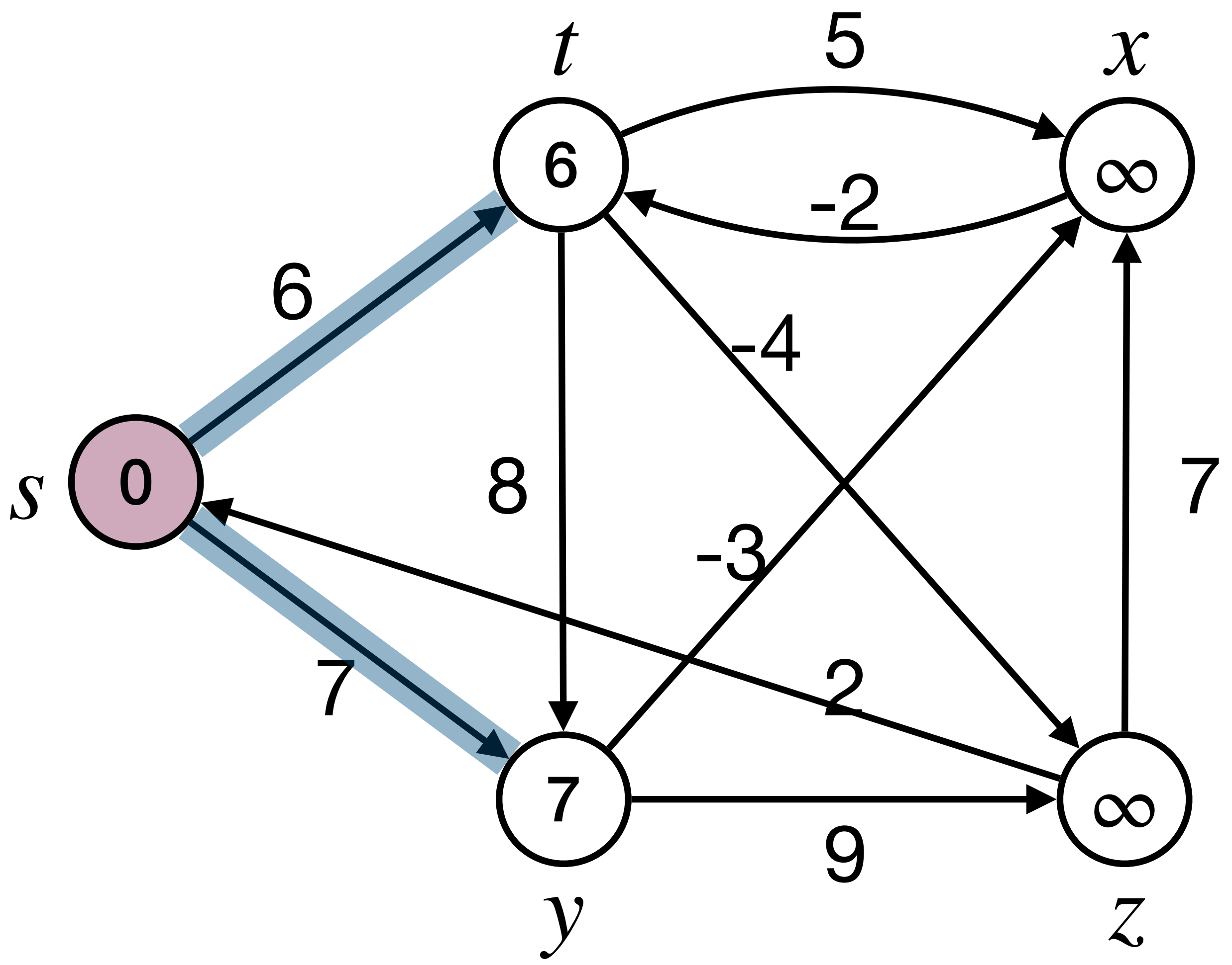
      $v.dist := u.dist + w(u, v)$

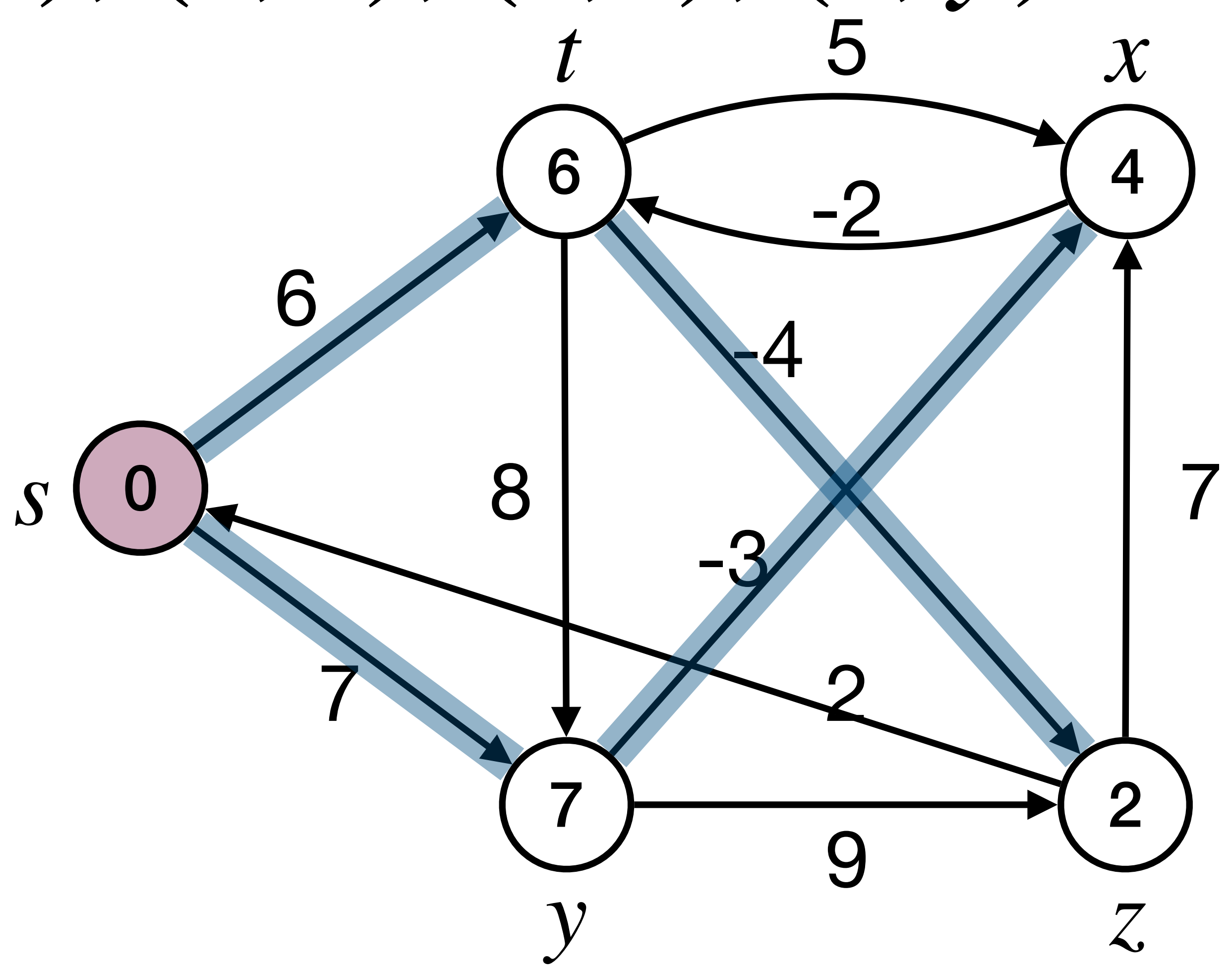
      $v.parent := u$

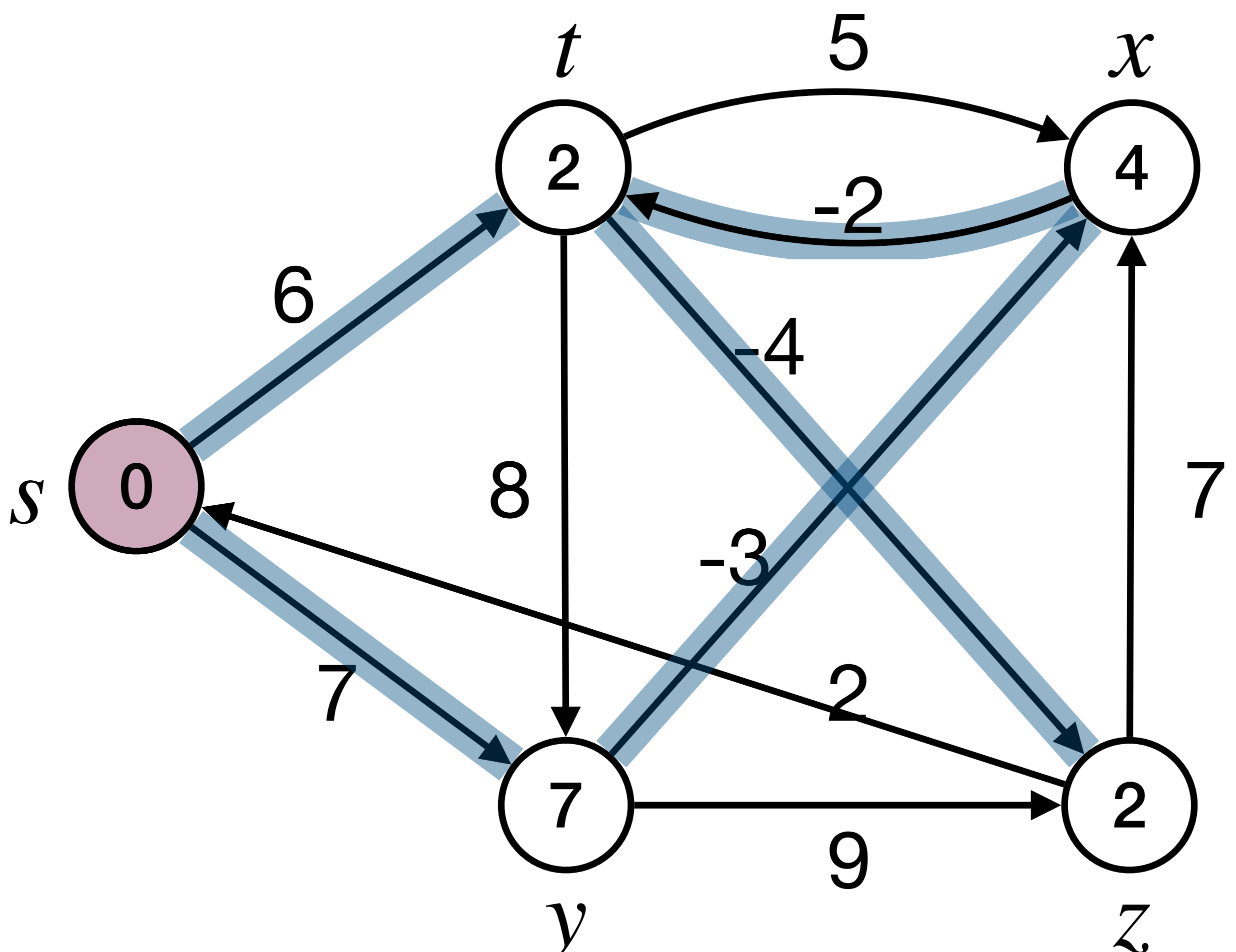
The Bellman-Ford Algorithm

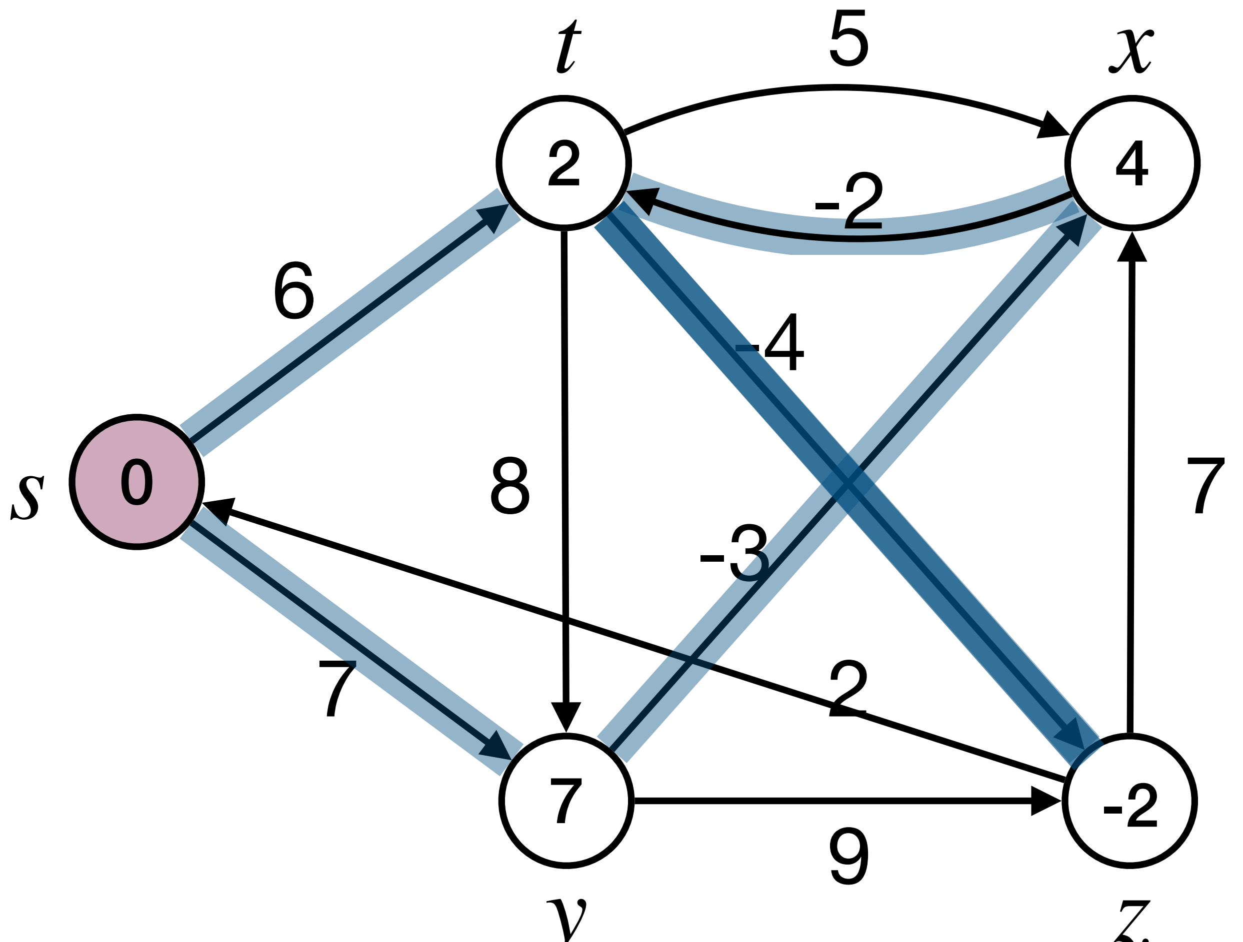
- Edge order: $(t, x), (t, y), (t, z), (x, t), (y, x), (y, z), (z, x), (z, s), (s, t), (s, y)$











The Bellman-Ford Algorithm

- What if the graph contains a negative cycle?
  - Then the **Observation 3** (any shortest path cannot contain a cycle.) does not hold!
  - It means that after $n-1$ repetitions of "Update all edges", some node v still has $v.dist > u.dist + w(u, v)$.

Bellman-Ford can also detect negative cycle!

BellmanFordSSSP(G, s):

for **each** u **in** V
    $u.dist := INF, u.parent := NIL$
$s.dist := 0$
repeat n - 1 times
    for **each** edge (u, v) **in** E
        if $v.dist > u.dist + w(u, v)$
            $v.dist := u.dist + w(u, v)$
            $v.parent := u$
for ***each*** *edge* (u, v) **in** E
    If $v.dist > u.dist + w(u, v)$
        return "negative circles"

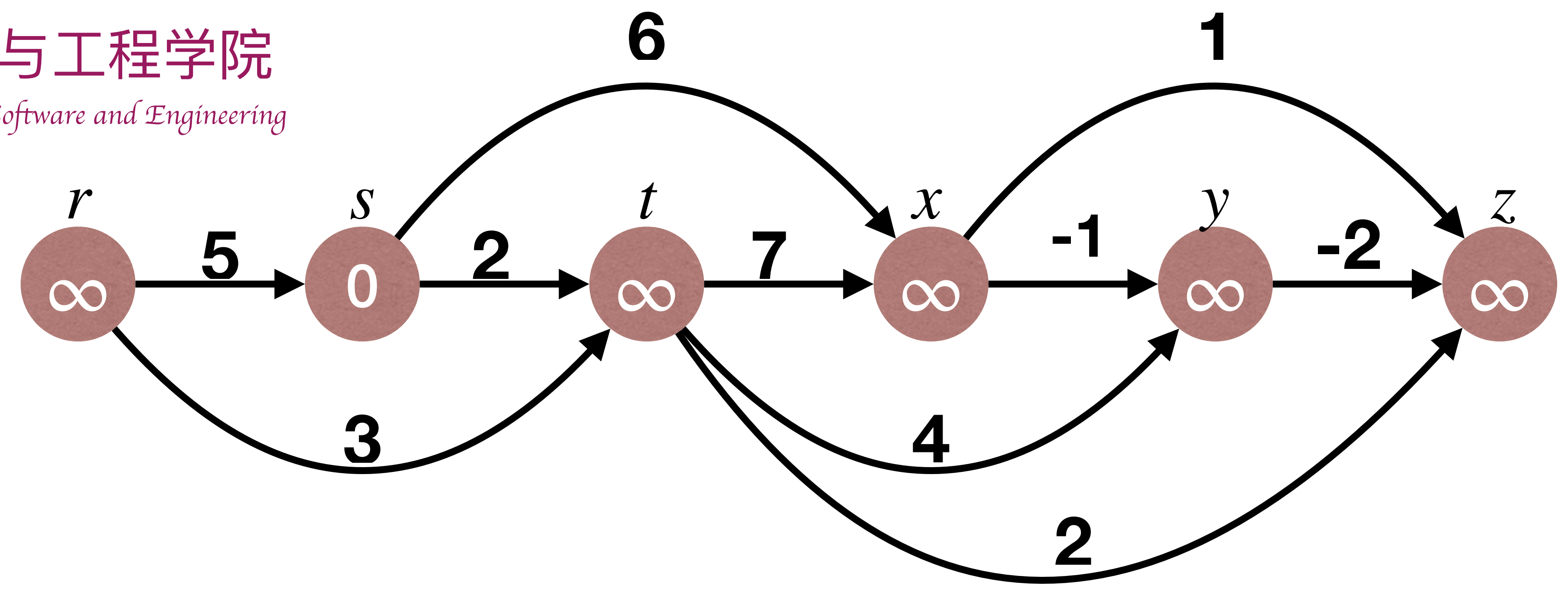
SSSP in DAG (with negative weights)

$O(m+n)$ time complexity

- Bellman-Ford still works, but we can be more efficient!
- Core idea of Bellman-Ford: perform a sequence of `Update` that includes every shortest path as a subsequence.
- Observation: in DAG, every path, thus every shortest path, is a subsequence in the topological order.

DAGSSSP(G,s):

for **each** u **in** V
    $u.dist := INF, u.parent := NIL$
$s.dist := 0$
Run DFS to obtain topological order
for **each** *node* u **in** *topological order*
    for **each** *edge* (u, v) **in** E
        if $v.dist > u.dist + w(u, v)$
            $v.dist := u.dist + w(u, v)$
            $v.parent := u$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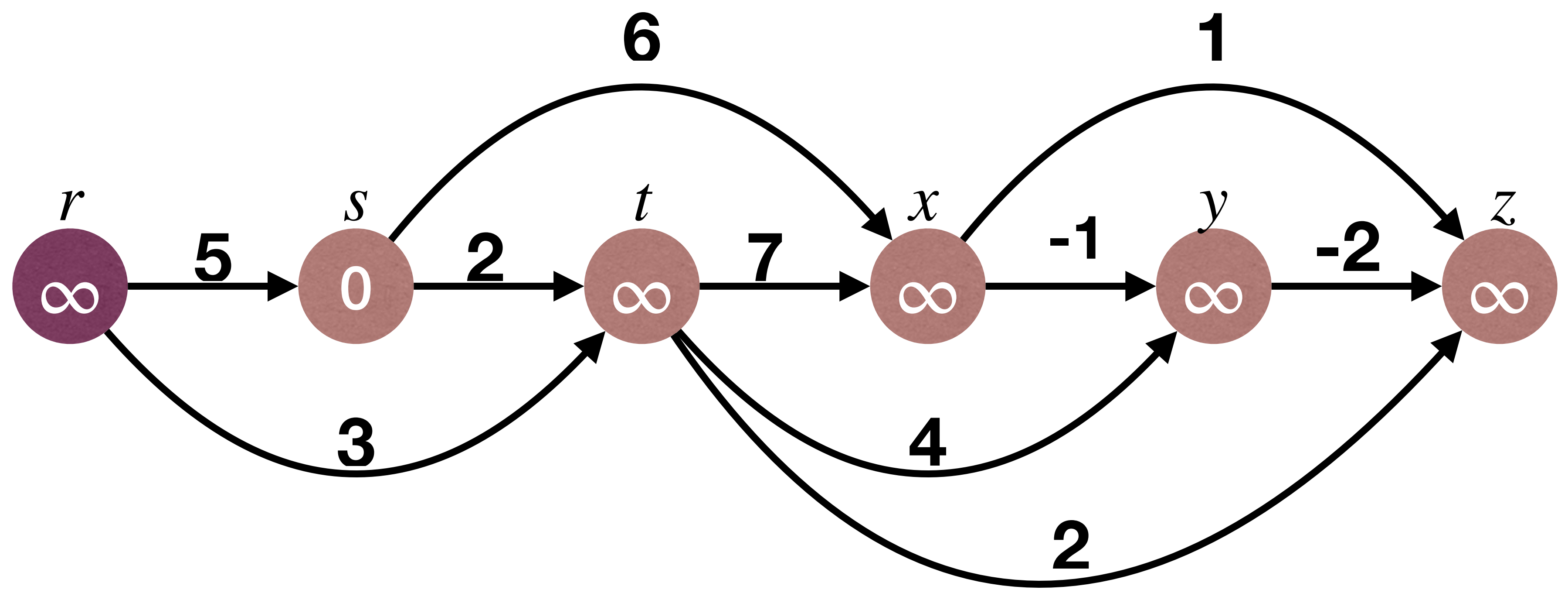
r s t x y z
5 2 7 -1 -2
∞ 0 ∞ ∞ ∞ ∞
3 4 2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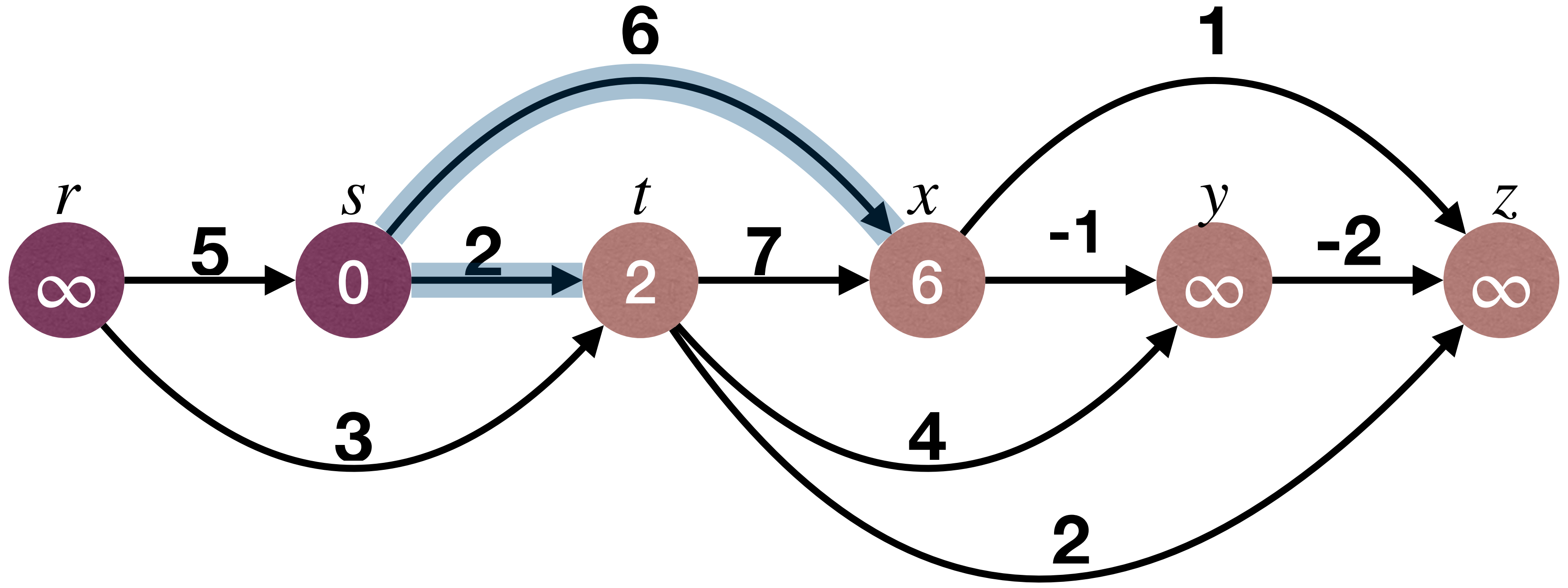
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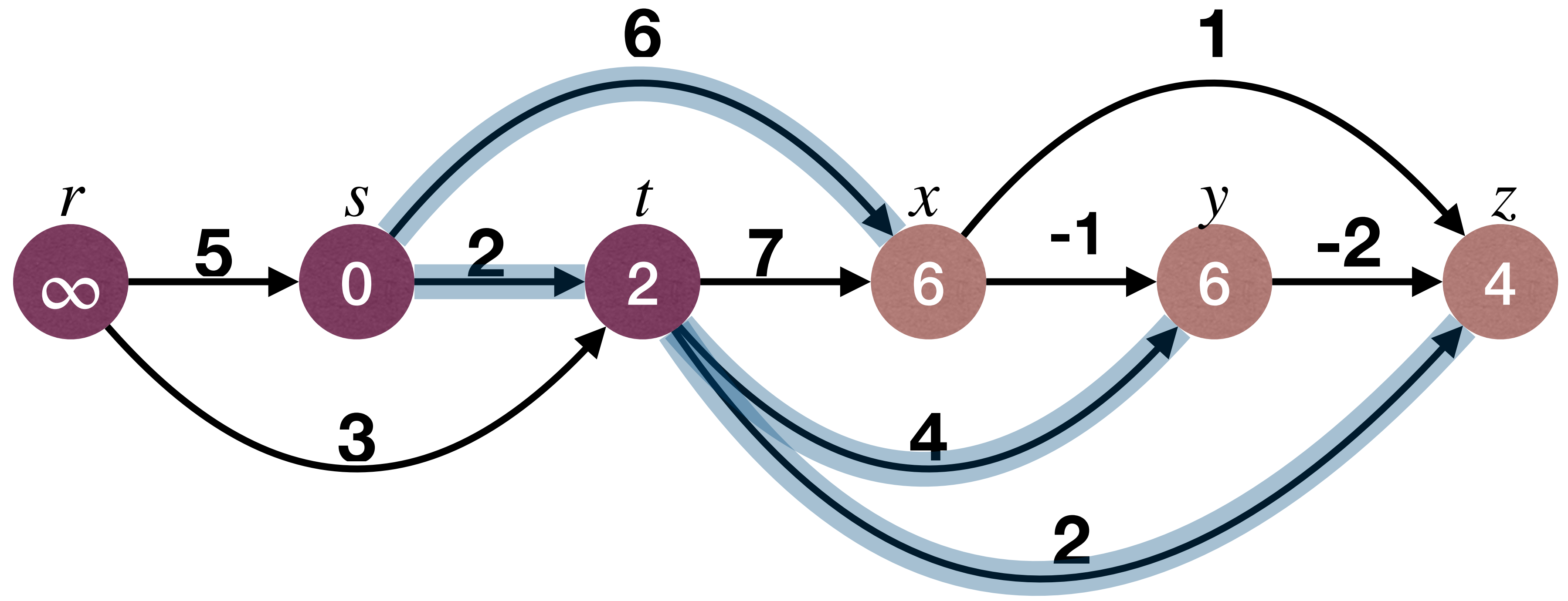
3 4 2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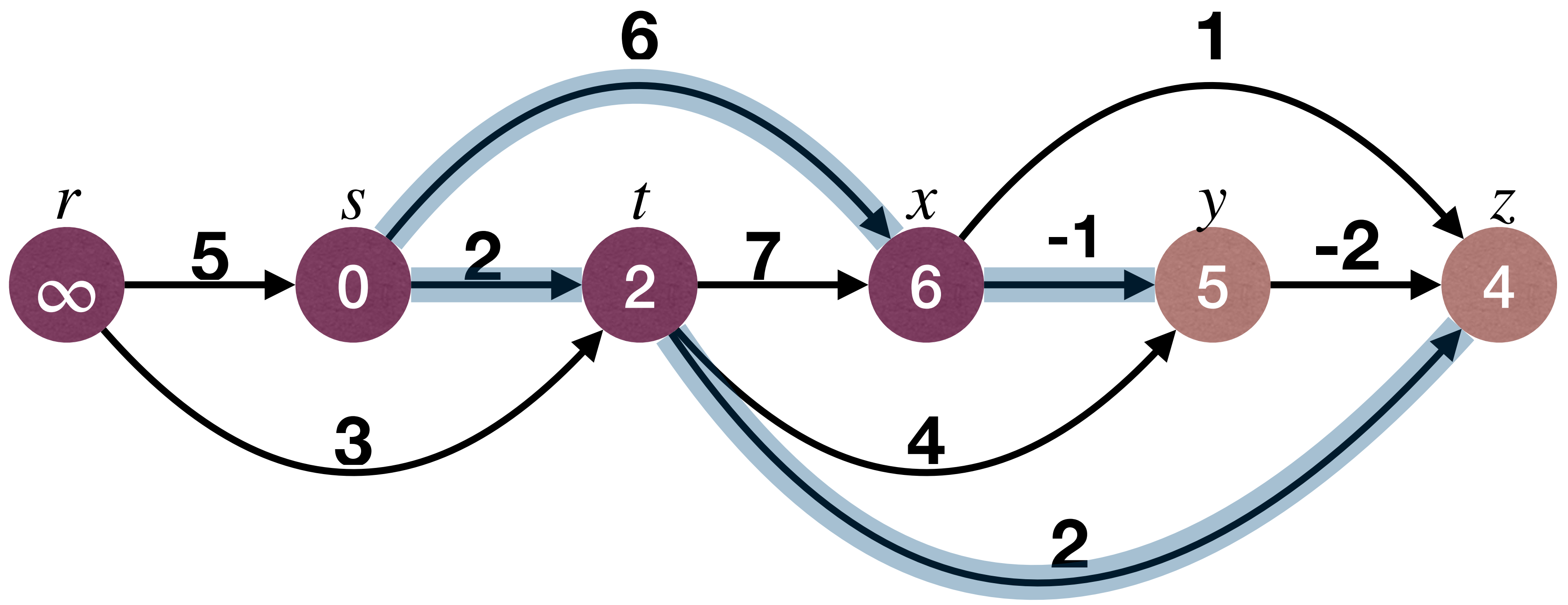
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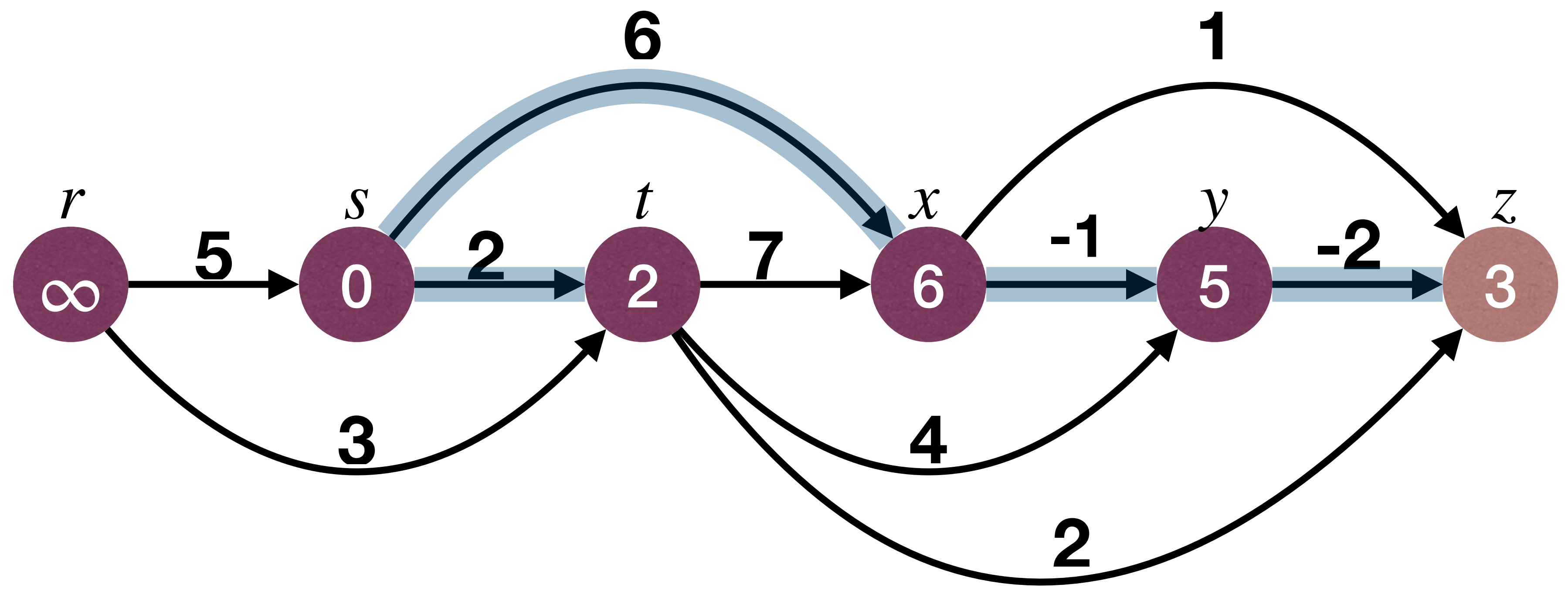
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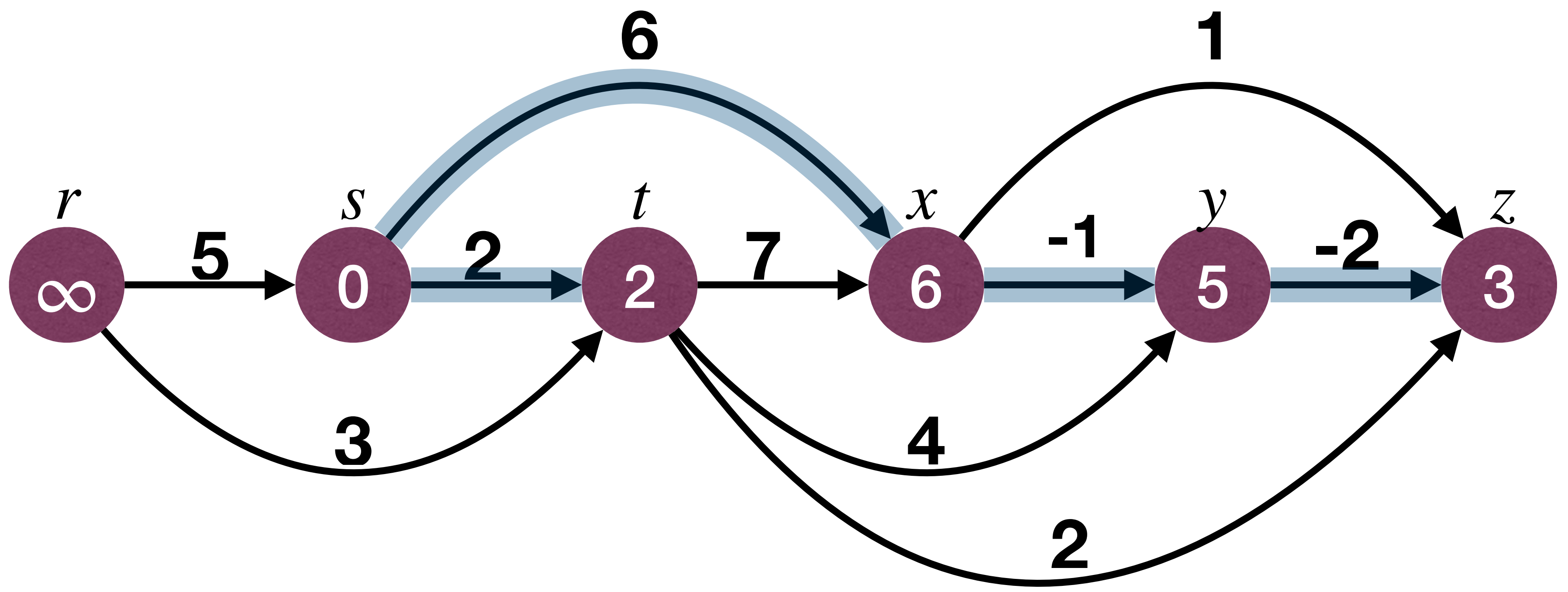
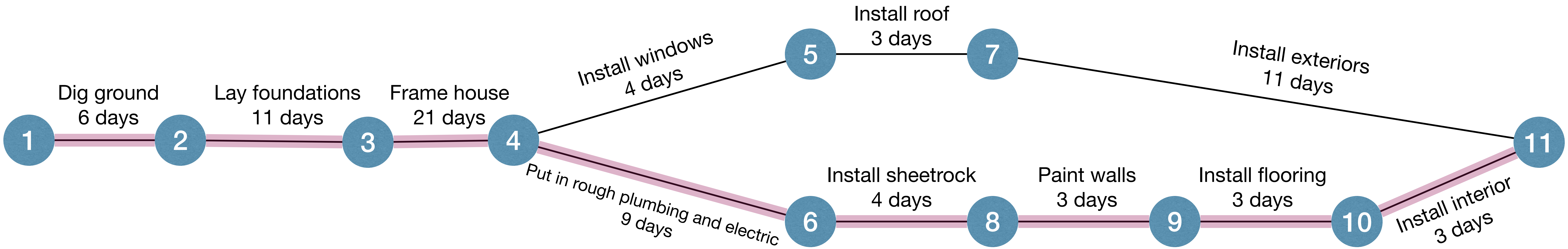

6 1
r s t x y z
5 2 7 -1 -2
∞ 0 2 6 5 3
3 4 2

Application of SSSP in DAG: Computing Critical Path

- Assume you want to finish a task that involves multiple steps. Each step takes some time. For some step(s), it can only begin after certain steps are done.
- These dependency can be modeled as a DAG. (PERT Chart)
- How fast can you finish this task?
- Equivalently, **longest path**, a.k.a. **critical path**, in the DAG?
- **Negate** edge weights and compute a **shortest** path.



Summary

- The SSSP Problem: Given a graph $G = (V, E)$ and a weight function w, given a source node s, find a shortest path from s to every node $v \in V$.

- **Case 1**: Unit weight graphs (directed or undirected): Simply use BFS. $O(n+m)$ runtime.

- **Case 2**: Arbitrary positive weight graphs (directed or undirected) : Dijkstra's algorithm. A greedy algorithm. $O((n+m)\log n)$runtime.

- **Case 3**: Arbitrary weight without cycle in directed graphs: `Update` in topological order. $O(n+m)$ runtime.

- **Case 4**: Arbitrary weight without negative cycle in directed graphs: Bellman-Ford algorithm. $\Theta(n(m+n))$ runtime, can detect negative cycle.

The shortest path problem has optimal substructure property.

`Update` **is a safe and helpful operation.**

Pathfinding*

(Shortest) Pathfinding*

- Given a graph $G = (V, E)$, how to find a (shortest) path from a source s to a destination t, preferably efficiently.

We could use BFS or Dijkstra.

But we could be **MUCH** faster!

Greedy Best-First Search

GreedyBFS(G, s, t):

$s.est\_to\_goal := heuristic(s,t)$

Build priority queue Q based on est_to_goal

while $!Q.empty()$

    $u := Q.ExtractMin()$

    for each *edge* (u,v) **in** E

        if $v \notin Q$

            $v.est\_to\_goal := heuristic(v,t)$

            $v.parent := u$

            $Q.Add(v)$

Does greedy BFS always generate correct answer?

- A (not necessarily accurate) *estimate* on the distance from v to t.
  - On 2D grid, we can set $heuristic(v,t) = ManhattanDist(v,t) = |v.x - t.x| + |v.y - t.y|$.

Greedy BFS does not always generate correct answer

PathfindingFramework(G, s, t):

for each *node u* **in** *V*

  $u.metric$:= $INFINITY$

$s.metric$:= $CalcMetric(s,s,t)$

Build priority queue Q based on metric

while $!Q.empty()$

  $u := Q.ExtractMin()$

  for each *edge* (u,v) **in** E

    $new\_metric := UpdateMetric(v, u, s, t)$

    if $v \notin Q$ **or** $new\_metric < v.metric$

      $v.metric := new\_metric$

      $v.parent := u$

      $Q.AddorUpdate(v)$

GreedyBFS: $est\_to\_goal(s, t)$

Dijkstra: $est\_to\_source(s,s) := 0$

GreedyBFS: $est\_to\_goal(v, t)$

Dijkstra:

$update\_est\_to\_source(v,u,s)$

$\min\{v.metric, u.metric + dist(u,v)\}$

a.k.a, $\min\{v.metric, dist(s, u) + dist(u,v)\}$

GreedyBFS is fast, but may be incorrect;
Dijkstra's algorithm is slower, but always correct;
Can we have an algorithm that is both fast and correct?

The A* algorithm

- For each node u:
  - $u.est\_to\_s$ maintains an (**over** or accurate) estimate of $dist(u,s)$, and this value changes during execution;
  - $u.est\_to\_t$ maintains an (**under** or accurate) estimate of $dist(u,t)$, and this value does not change during execution.
  - Use $u.est\_to\_s + u.est\_to\_t$ as the metric to guide the search!
- Usually set to the straight-line distance between u and t.

AStarPathfinding(G, s, t):

for each *node* u **in** V

  $u.est\_to\_s := INFINITY$

  $u.est\_to\_t := heuristic(u,t)$

  $u.metric := u.est\_to\_s + u.est\_to\_t$

$s.est\_to\_s := 0$, $s.metric := s.est\_to\_s + s.est\_to\_t$

Build priority queue Q based on metric

while $!Q.empty()$

  $u := Q.ExtractMin()$

  for each *edge* (u,v) **in** E

    if $v \notin Q$ **or** $v.est\_to\_s > u.est\_to\_s + dist(u, v)$

      $v.est\_to\_s := u.est\_to\_s + dist(u, v)$

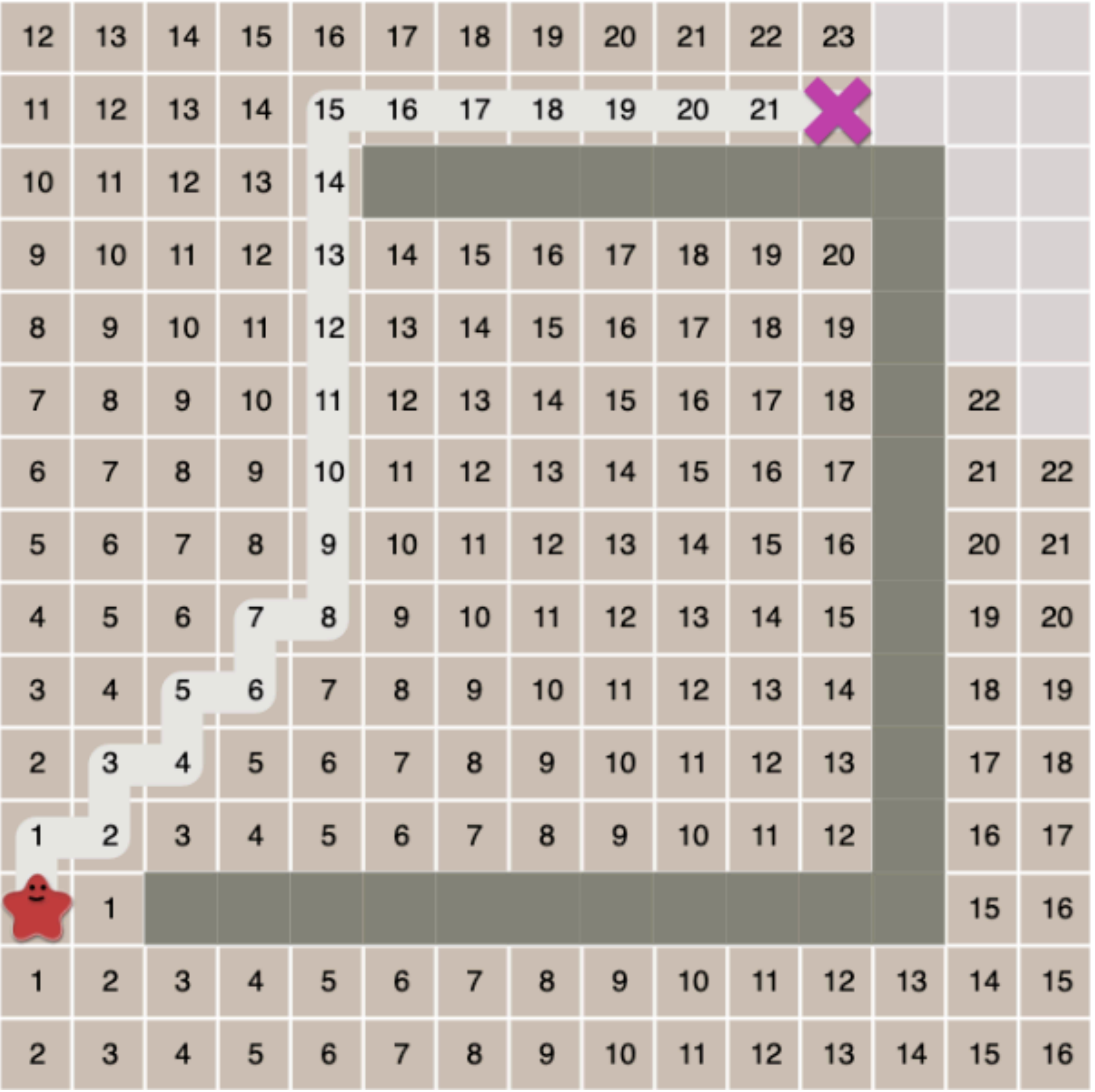
      $v.metric := v.est\_to\_s + v.est\_to\_t$

      $v.parent := u$

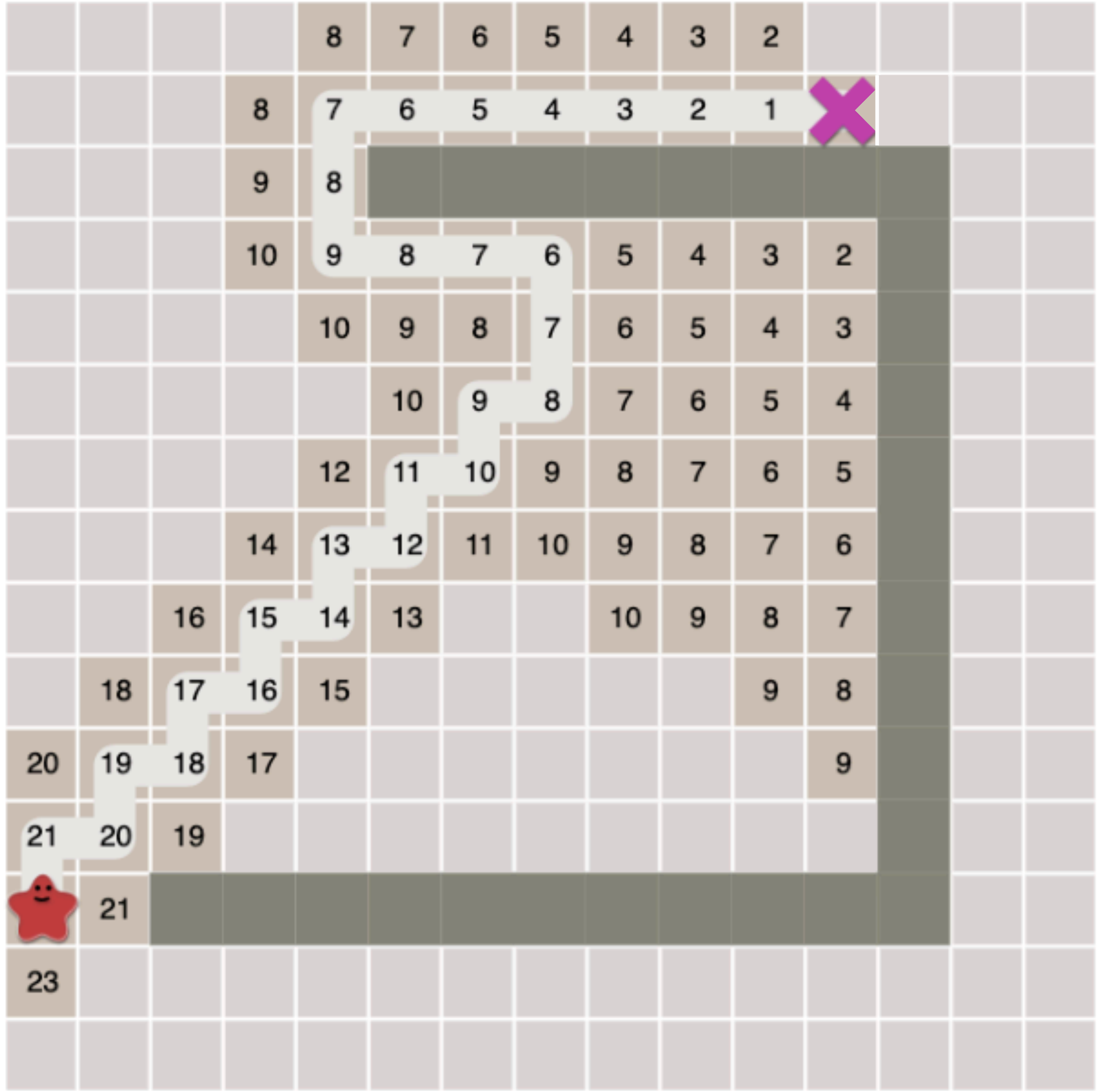
      $Q.Add(v)$

The A* algorithm

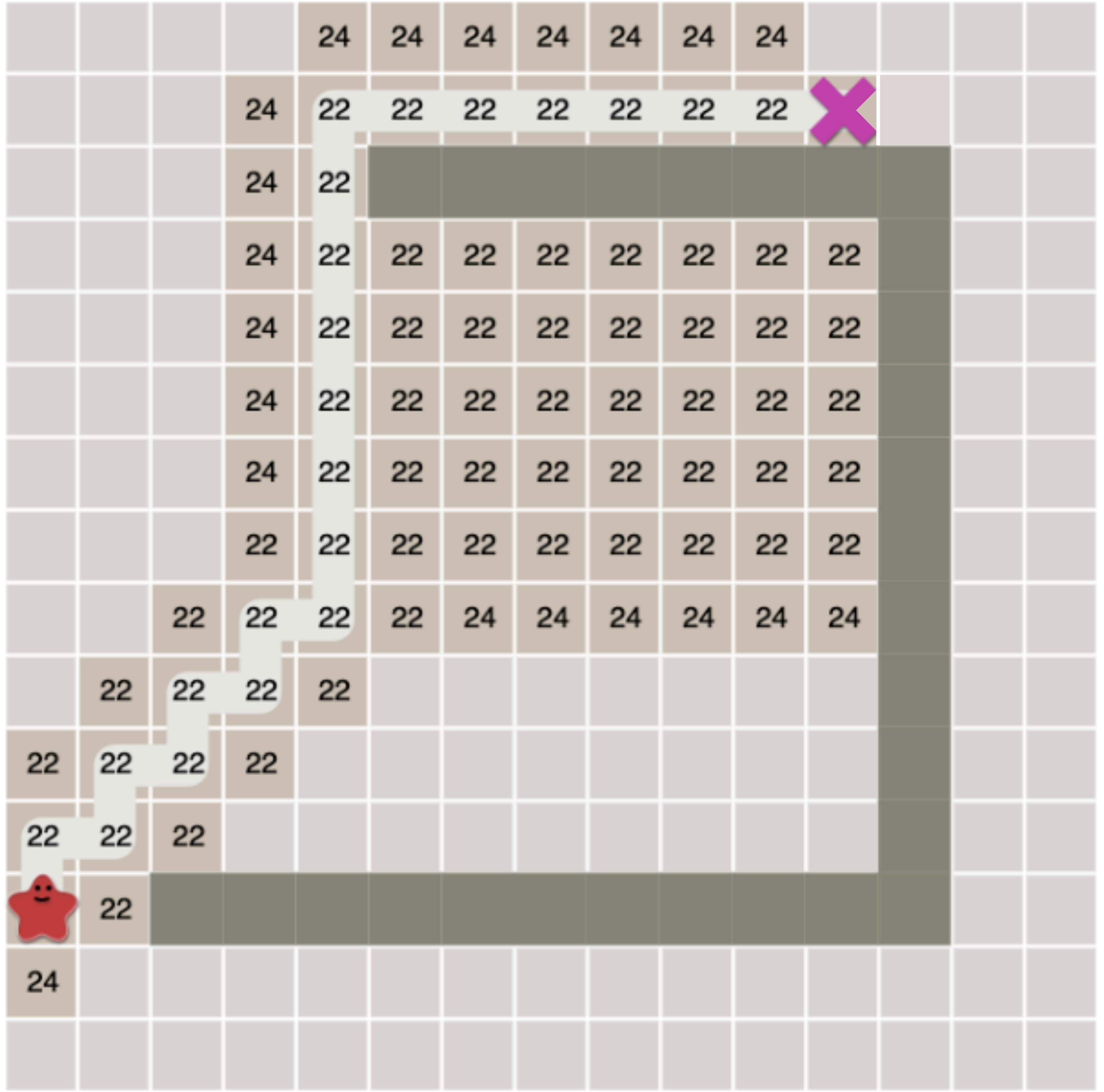
Dijkstra's Algorithm



Greedy Best-First



A* Search



f : metric, h: estimate to goal

The A* algorithm

- Correctness of the A* algorithm?
  - It is correct as long as $u.est\_to\_t \leq dist(u,t)$ always hold.
- Time complexity of the A* algorithm?
  - More complicated as a node may be added to the queue multiple times.
  - In AI community, it is normally considered to be $O(b^d)$, where b is the branching factor (the average number of successors per state), and d is the depth of the solution (the shortest path).
  - The heuristic function has a major effect on the practical performance of A* search, since a good heuristic allows A* to prune away many of the b^d nodes.

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Further reading

- [CLRS] Ch.24 (excluding 24.4)
- [DPV] Ch.4
- [Erickson] Ch.8
- Refer to https://www.redblobgames.com/pathfinding/a-star/introduction.html if you want to know more about A* algorithm