全源最短路径
All-Pairs Shortest Path

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The slides are mainly adapted from the original ones shared by Chaodong Zheng and Kevin Wayne. Thanks for their supports!

SSSP and APSP

- Single-Source Shortest Paths (SSSP) Problem:
  - Given a graph $G = (V, E)$ and a weight function w, given a source node s, find a shortest path from s to every $u \in V$.
- All-Pairs Shortest Paths (APSP) Problem:
  - Given a graph $G = (V, E)$ and a weight function w, for every pair $(u, v) \in V \times V$, find a shortest path from u to v.

APSP from multiple SSSP

- Straightforward solution for APSP: For each $v \in V$, execute SSSP algorithm once!

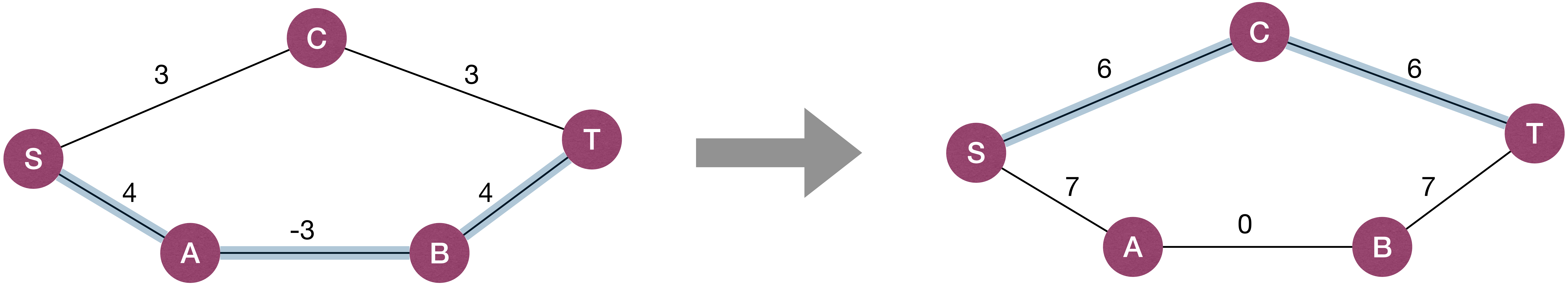
	SSSP	APSP
BFS (Unit-weight graphs)	$O(n+m)=O(n^2)$	$O(n^3)$
Dijkstra (Positive-weight graphs)	$O((n+m)\lg n)=O(n^2\lg n)$ (Using binary heap for priority queue)	$O(n^3\lg n)$
Bellman-Ford (Arbitary-weight Directed)	$O(nm)=O(n^3)$	$O(n^4)$
Topological Sort Variant (Arbitrary-weight DAG)	$O(n+m)=O(n^2)$	$O(n^3)$

APSP from multiple SSSP

- Positive-weight Graphs: Repeating Dijkstra gives $O(n^3 \lg n)$.
- Arbitrary-weight Graphs: Repeating Bellman-Ford gives $O(n^4)$.
- Faster algorithms for arbitrary-weight graphs?
  - **Intuition**: modify edge weights without changing shortest path, so that Dijkstra's algorithm can work.

APSP from multiple SSSP

- **Intuition**: modify edge weights *without* changing shortest path, so that Dijkstra's algorithm can work.
  - Add $\max\{-1 \cdot w(u,v)\}$ to each edge?
- NO! Shortest paths may change!
- Given (u,v), different paths may change by different amount!



APSP from multiple SSSP

- Faster algorithms for arbitrary-weight graphs?
  - **Intuition**: modify edge weights without changing shortest path, so that Dijkstra's algorithm can work.
  - **Requirement**: $\hat{w}(u \overset{p_1}{\rightsquigarrow} v) > \hat{w}(u \overset{p_2}{\rightsquigarrow} v) \iff w(u \overset{p_1}{\rightsquigarrow} v) > w(u \overset{p_2}{\rightsquigarrow} v)$

    new weight of path
  - **Or alternatively**, for every path from u to v, $\hat{w}$ changes it by the **same** amount:
    - Let the $\hat{w}(u, v) = h(u) + w(u, v) - h(v)$

      new weight of edge
    - Imagine $h(u)$ is **entry gift** and $h(v)$ is **exit tax** for traveling through (u, v).

APSP from multiple SSSP

- $\hat{w}(u \overset{p_1}{\rightsquigarrow} v) = \hat{w}(u \to x_1 \to \ldots \to x_k \to v) = \hat{w}(u \to x_1) + \ldots + \hat{w}(x_k \to v)$

$= \left(h(u) + w(u \to x_1) - h(x_1)\right) + \left(h(x_1) + w(x_1 \to x_2) - h(x_2)\right) + \ldots + \left(h(x_{k-1}) + w(x_{k-1} \to x_k) - h(x_k)\right) + \left(h(x_k) + w(x_k \to v) - h(v)\right)$

$= h(u) + w(u \to x_1) + \ldots + w(x_k \to v) - h(v)$

$= h(u) + w(u \to x_1 \to \ldots \to x_k \to v) - h(v) = h(u) + w(u \overset{p_1}{\rightsquigarrow} v) - h(v)$

APSP from multiple SSSP

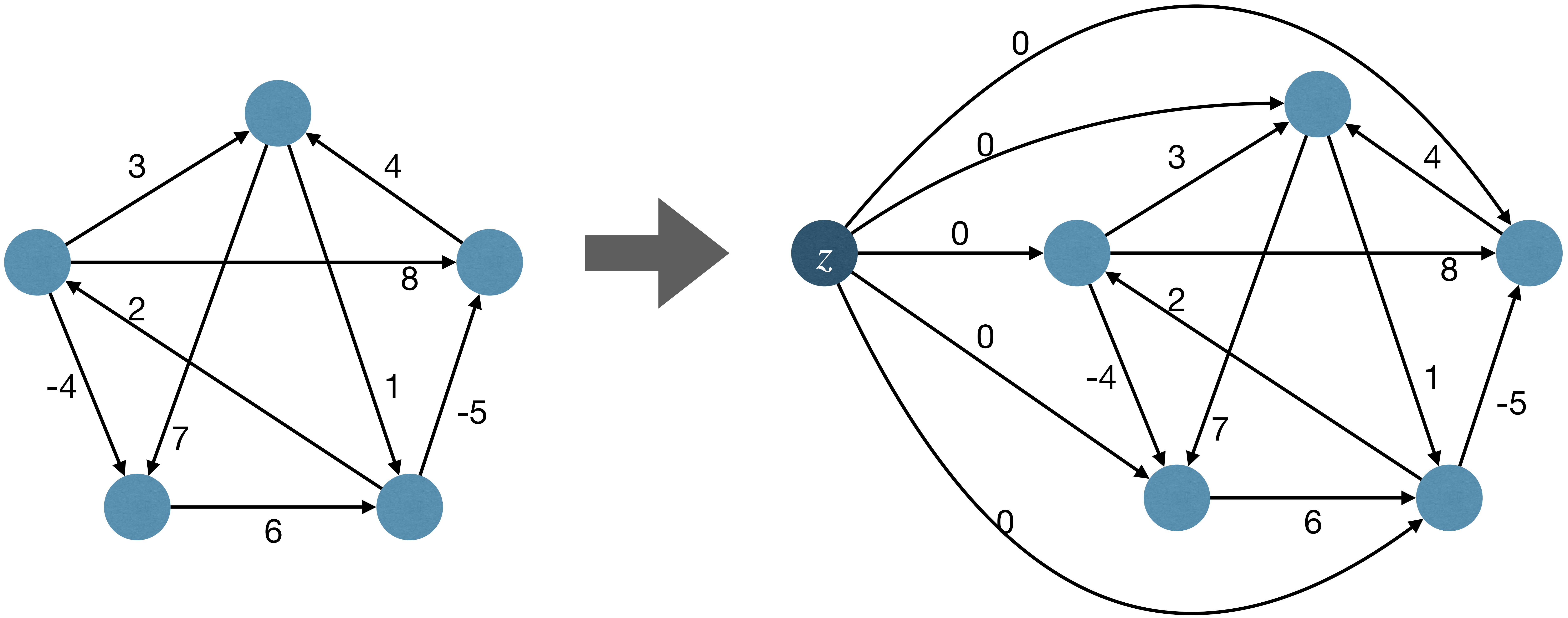
- Since we need $\hat{w}(u,v) = h(u) + w(u,v) - h(v) \geq 0$ (for Dijkstra algorithm)
- Just let $h(u) = dist(z,u)$ for some fixed $z \in V$, then
  $\hat{w}(u,v) = dist(u) + w(u,v) - dist(v) \geq 0$

The shortest path from z to v must be "smaller than" the shortest path from z to u add the edge from u to v.

  - But it is possible that we cannot find such z that reaches every node.

APSP from multiple SSSP

- Add node z that goes to every node in G with a weight 0 edge.
  - $H = (V \cup \{z\}, E \cup \{(z,x) \mid x \in V\})$ with $w(z,x) = 0$



APSP from multiple SSSP

- Re-weight edges:
  $\hat{w}(u,v) = dist(u) + w(u,v) - dist(v) \geq 0$
  - For node pairs in G, addition of z does not create new shortest path.
  - For node pairs in G, a path is shortest under w iff this path is shortest under $\hat{w}$.

JohnsonAPSP(G,s):

Create $H := (V+\{z\}, E+\{(z, v) \mid v \in V\})$ *with* $w(z,v) = 0$

Bellman-FordSSSP(H,z) *to obtain* $dist_H$

for each *edge* (u, v) **in** $H.E$

  $w'(u,v) := dist_H(z,u) + w(u,v) - dist_H(z,v)$

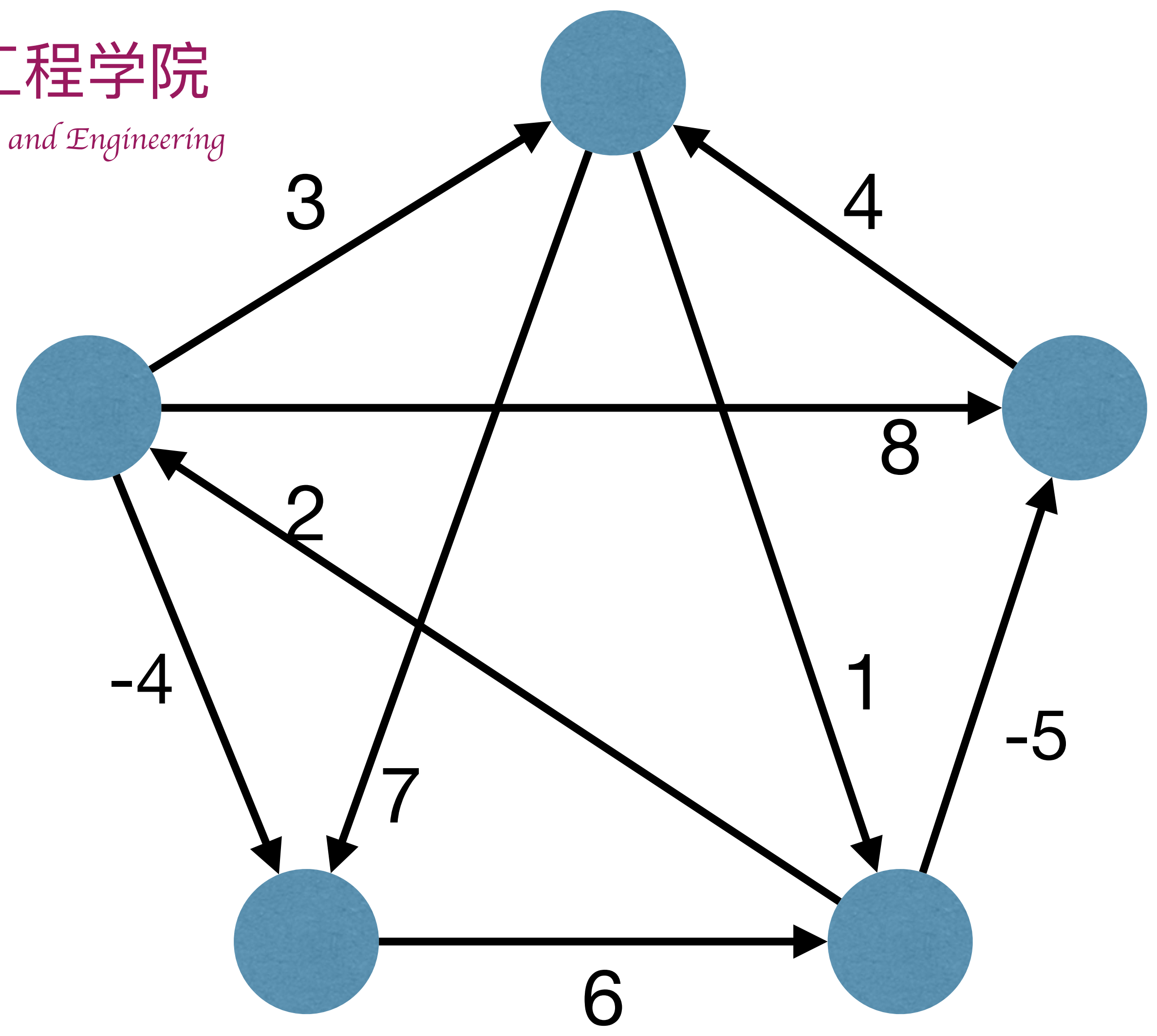
for each *node* u **in** $G.V$

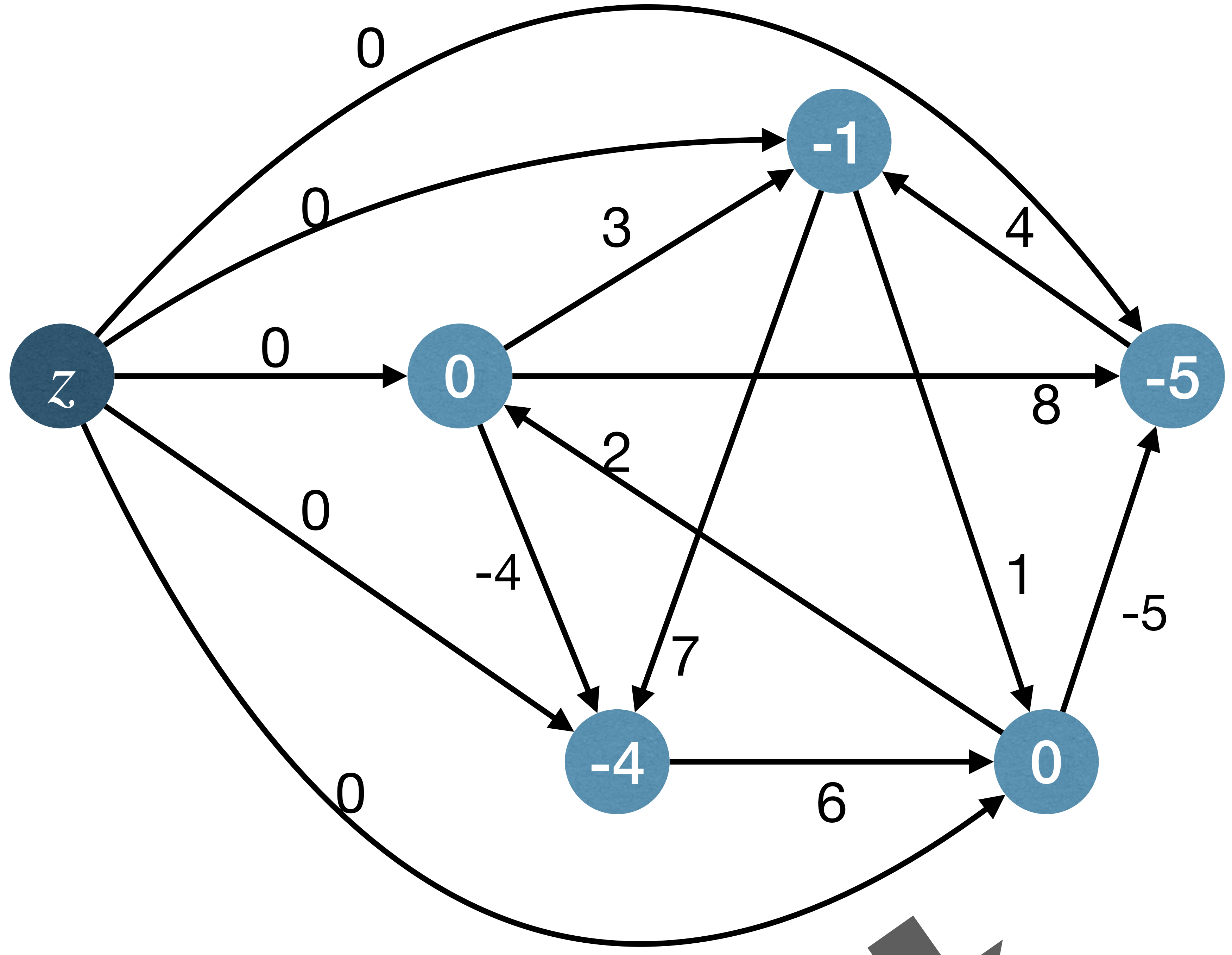
  DijkstraSSSP(G,u) *with* w' *to obtain* $dist_{G,w'}$

  for each *node* v **in** $G.V$

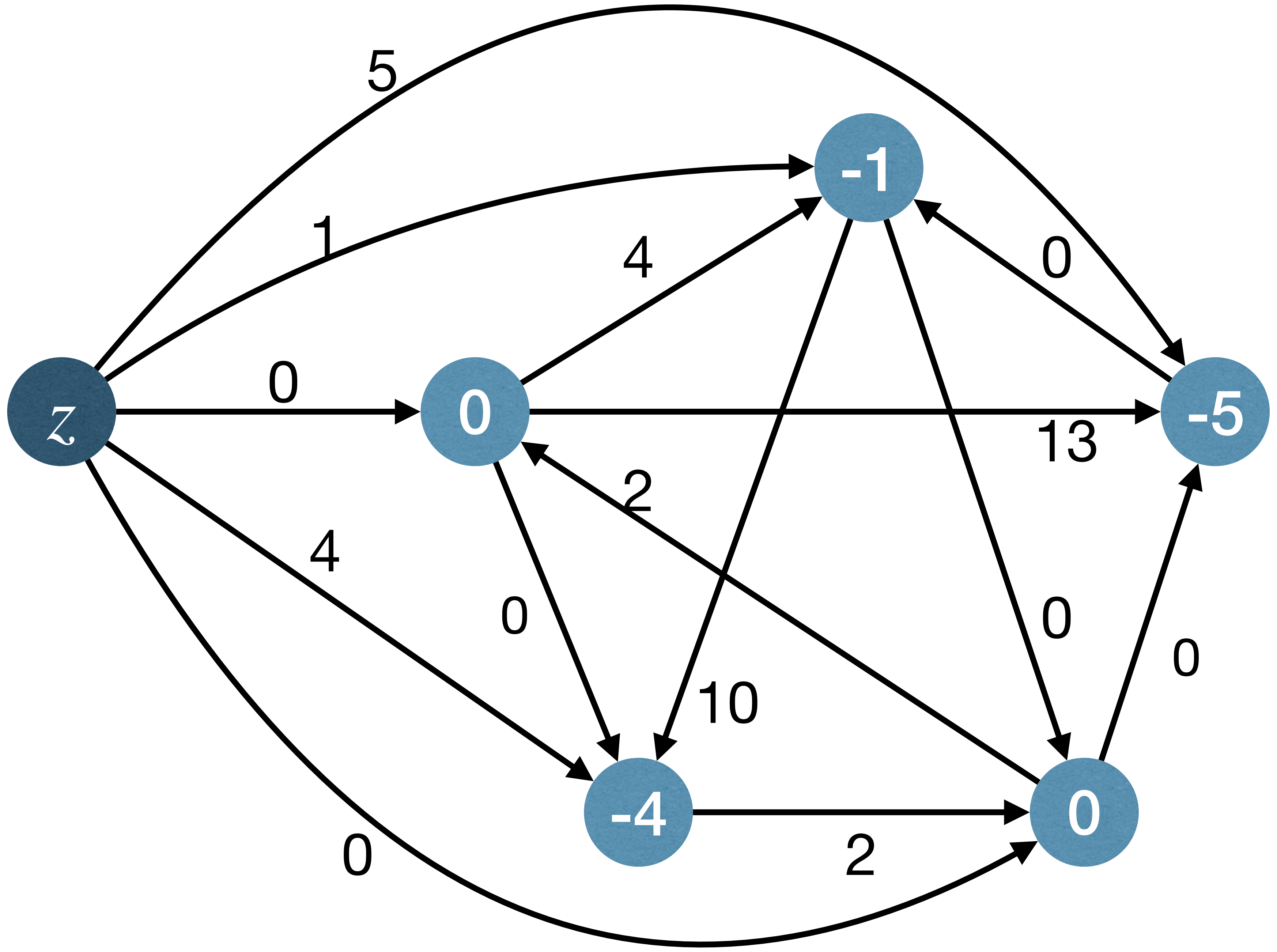
    $dist_G(u,v) := dist_{G,w'}(u,v) + dist_H(z,v) - dist_H(z,u)$

Proposed by Donald Bruce Johnson

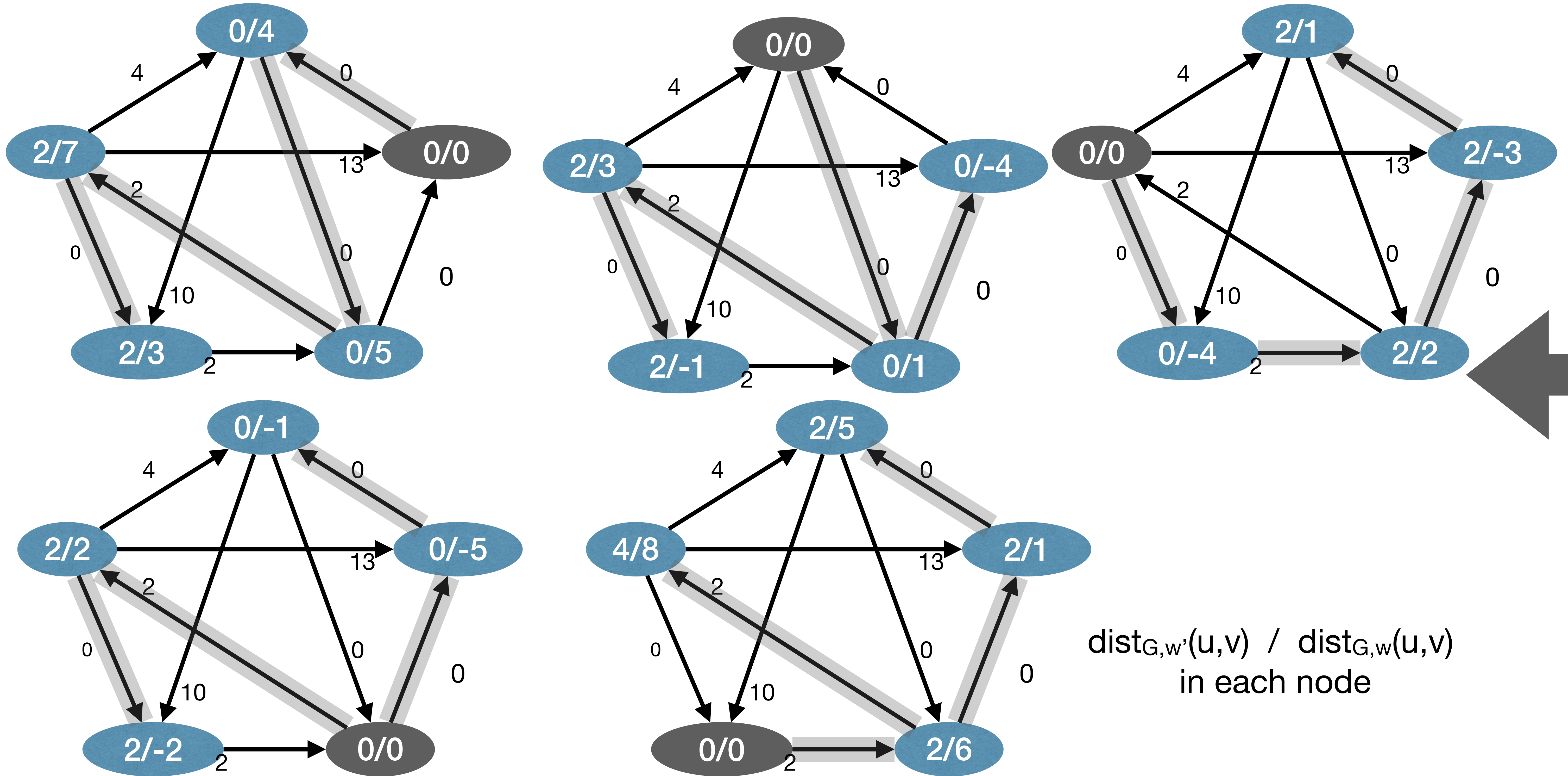




Add a new node z
Compute each $h(u)$



re-weighting each edge
By $\hat{w}(u,v) = h(u) + w(u,v) - h(v)$



$\text{dist}_{G,w'}(u,v)$ / $\text{dist}_{G,w}(u,v)$
in each node

APSP from multiple SSSP

- Johnson's algorithm combines Dijkstra and Bellman-Ford, resulting a runtime of $O(n^3 \lg n)$, for arbitrary weight graphs.

JohnsonAPSP(G,s):

Create H := $(V+\{z\}, E+\{(z, v) \mid v \in V\})$ *with* $w(z,v) = 0$

Bellman-FordSSSP(H,z) to obtain $dist_H$

for each *edge* (u, v) **in** $H.E$

  $w'(u,v) := dist_H(z,u) + w(u,v) - dist_H(z,v)$

for each *node* u **in** $G.V$

  DijkstraSSSP(G,u) with w' to obtain $dist_{G,w'}$

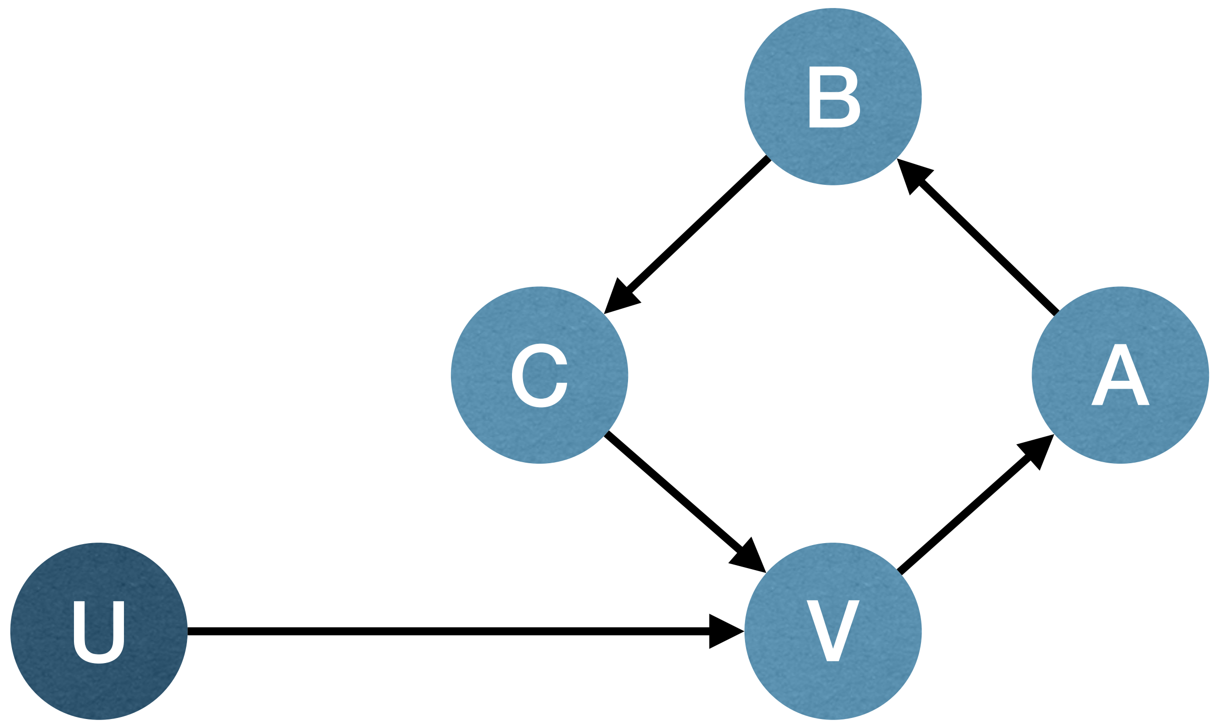
  for each *node* v **in** $G.V$

    $dist_G(u,v) := dist_{G,w'}(u,v) + dist_H(z,v) - dist_H(z,u)$

Floyd-Warshall Algorithm

APSP via Recursion

- $dist(u,v) = \begin{cases} 0 & if\ u = v \\ \min_{(x,v)\in E}\{dist(u,x) + w(x,v)\} & otherwise \end{cases}$
- This recurrence is correct, but it does not lead to a recursive algorithm directly!
    - Cycle in the graph can make the recursion never ends!



APSP via Recursion

- Introduce an additional parameter in the recurrence:
  - $dist(u, v, l)$: shortest path from u to v that uses at most l edges.
- $$dist(u, v) = \begin{cases} 0 & if\ l = 0\ and\ u = v \\ \infty & if\ l = 0\ and\ u \neq v \\ \min \left\{ \begin{array}{l} dist(u, v, l-1) \\ \min\limits_{(x,v)\in E} \{dist(u, x, l-1) + w(x, v)\} \end{array} \right\} & otherwise \end{cases}$$

APSP via Recursion

- $$dist(u,v) = \begin{cases} 0 & if\ l = 0\ and\ u = v \\ \infty & if\ l = 0\ and\ u \neq v \\ \min \left\{ \begin{array}{l} dist(u,v,l-1) \\ \min\limits_{(x,v)\in E} \{dist(u,x,l-1) + w(x,v)\} \end{array} \right\} & otherwise \end{cases}$$

- Evaluate this recurrence easily in a "**bottom-up**" fashion!
  - $dist(\cdot,\cdot,0)$ are easy to compute, given input graph.
  - $dist(\cdot,\cdot,1)$ are easy to compute, if $dist(\cdot,\cdot,0)$ are known.
  - $dist(\cdot,\cdot,l+1)$ are easy to compute, if $dist(\cdot,\cdot,l)$ are known.
  - $dist(\cdot,\cdot,n-1)$ are what we want!

Don't always need a recursive algorithm to evaluate recurrence, often an iterative alternative exists.

APSP via Recursion

RecursiveAPSP(G):

for **each** *pair* (u,v) **in** $V*V$

  if $u = v$ then $dist[u,v,0] := 0$

  else $dist[u,v,0] := INF$

for $l := 1$ **to** $n - 1$

  for **each** *node* u

    for **each** *node* v

      $dist[u,v,l] := dist[u,v,l-1]$

      for **each** *edge* (x,v) going to v

        if $dist[u,v,l] > dist[u,x,l-1] + w(x,v)$

          $dist[u,v,l] := dist[u,x,l-1] + w(x,v)$

Time complexity:
$O(n^4)$

Can we do better?

APSP via Recursion

- $$dist(u,v) = \begin{cases} 0 & \text{if } l = 0 \text{ and } u = v \\ \infty & \text{if } l = 0 \text{ and } u \neq v \\ \min \left\{ \begin{matrix} dist(u,v,l-1) \\ \min_{(x,v)\in E} \{dist(u,x,l-1) + w(x,v)\} \end{matrix} \right\} & \text{otherwise} \end{cases}$$
- This recursion is like "1 and $l-1$ split" in divide-and-conquer. How about "$l/2$ and $l/2$ split"?
- $$dist(u,v,l) = \begin{cases} w(u,v) & \text{if } l = 1 \text{ and } (u,v) \in E \\ \infty & \text{if } l = 1 \text{ and } (u,v) \notin E \\ \min_{x\in V}\{dist(u,x,l/2) + dist(x,v,l/2)\} & \text{otherwise} \end{cases}$$
- Start with $dist(\cdot,\cdot,1)$, then double l each time, until $2^{\lceil \lg n \rceil}$.

APSP via Recursion

<u>FasterRecursiveAPSP(G):</u>

for each *pair* (u,v) **in** $V*V$

  if (u, v) **in** E **then** $dist[u,v,1] := w(u, v)$

  else $dist[u,v,1] := INF$

for $i := 1$ **to** $\lceil \lg n \rceil$

  for each *node* u

    for each *node* v

      $dist[u,v,2^i] := INF$

      for each *node* x

        if $dist[u,v,2^i] > dist[u,x,2^{i-1}] + dist[x,v,2^{i-1}]$

          $dist[u,v,2^i] := dist[u,x,2^{i-1}] + dist[x,v,2^{i-1}]$

Time complexity:
$O(n^3 \lg n)$

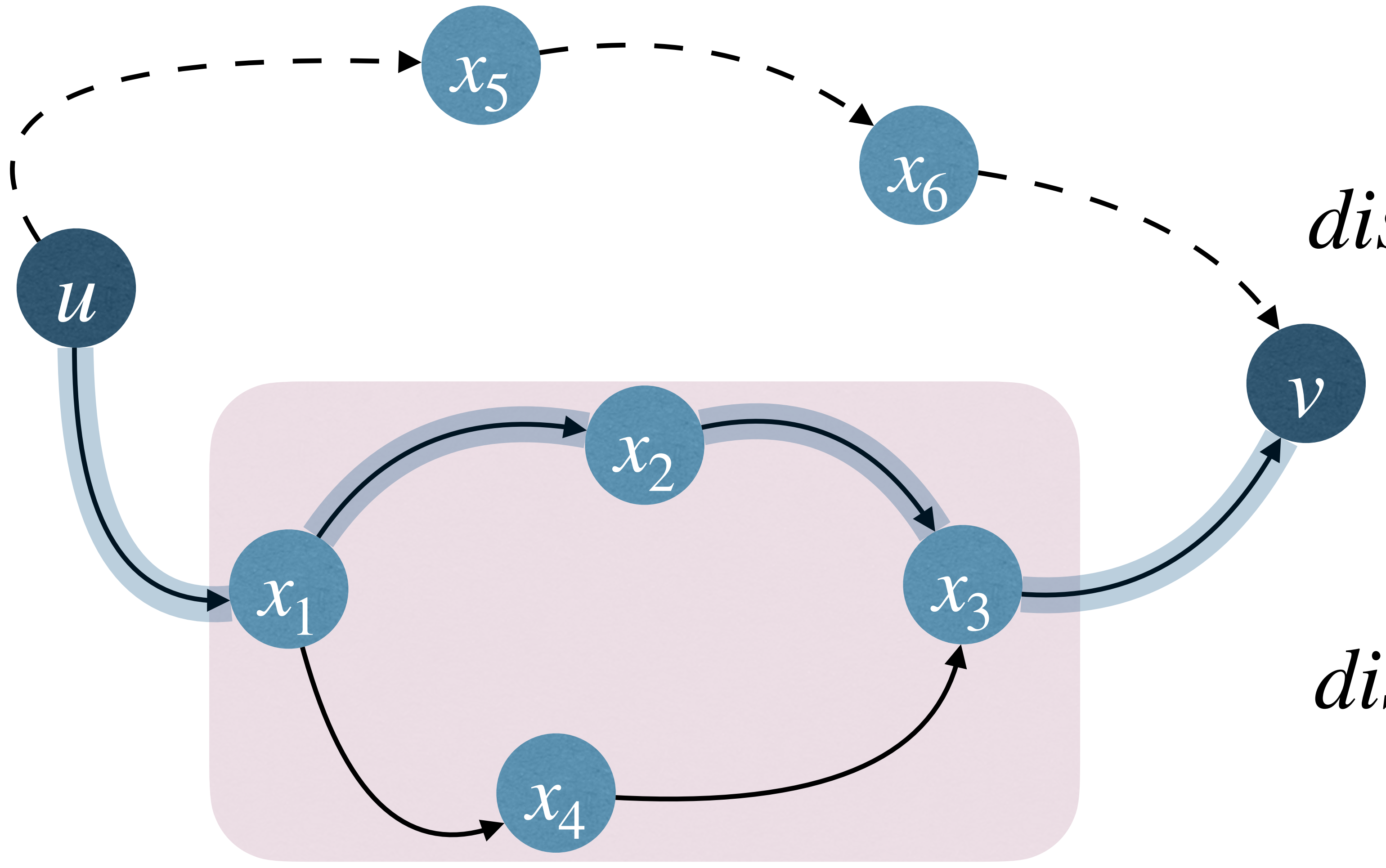
Can this approach be better?

APSP via Recursion

- **Strategy:** recuse on the *set of node* the shortest paths use.(Previous algorithms recuse on number of edges the shortest paths use.)

- Number the vertices arbitrarily: $x_1, x_2, \ldots, x_n$; Define $V_r = \{x_1, x_2, \ldots, x_r\}$ to be the set of vertices numbered at most r.

- Define $dist(u, v, r)$ be length of shortest path from u to v, s.t. only nodes in V_r can be ***intermediate*** nodes in paths. Let $\pi(u, v, r)$ be such a shortest path.



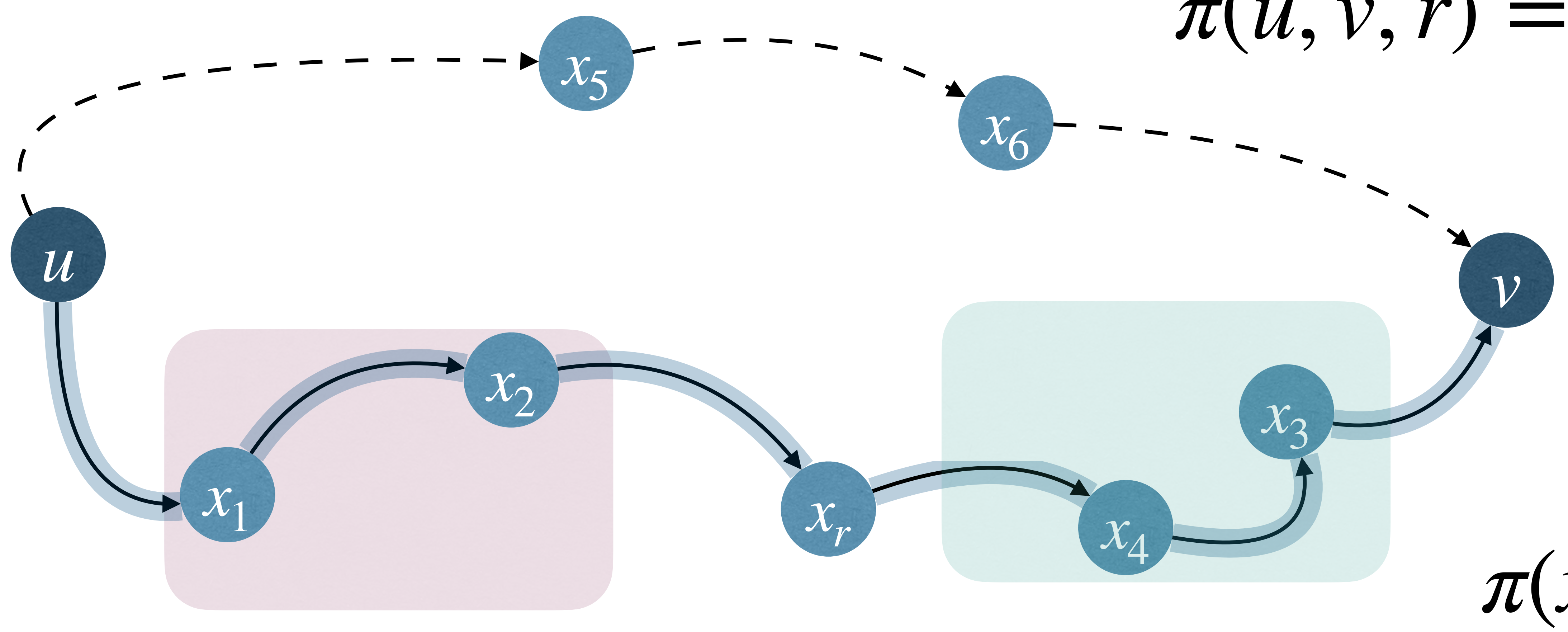
$dist(u, v) = dist(u, v, 6) = w(u \to x_5 \to x_6 \to v)$

$dist(u, v, 4) = w(u \to x_1 \to x_2 \to x_3 \to v)$

APSP via Recursion

- **Observation:** either $\pi(u, v, r)$ goes through x_r or not.
- **Latter case**: $\pi(u, v, r) = \pi(u, v, r-1)$
- **Former case:** $\pi(u, v, r) = \pi(u, x_r, r) + \pi(x_r, v, r) = \pi(u, x_r, r-1) + \pi(x_r, v, r-1)$

$\pi(u, v, r) = \pi(u, v, r-1)$



$\pi(u, x_r, r-1)$

$\pi(x_r, v, r-1)$

The Floyd-Warshall Algorithm

- $$dist(u,v,r) = \begin{cases} w(u,v) & \text{if } r = 0 \text{ and } (u,v) \in E \\ \infty & \text{if } r = 0 \text{ and } (u,v) \notin E \\ \min \begin{Bmatrix} dist(u,v,r-1) \\ dist(u,x_r,r-1) + dist(x_r,v,r-1) \end{Bmatrix} & \text{otherwise} \end{cases}$$

Bernard Roy

Robert W. Floyd

Stephen Warshall

FloydWarshallAPSP(G):

```
for each pair (u,v) in V*V
    if (u, v) in E  then dist[u,v, 0] := w(u, v)
    else dist[u,v,0] := INF
for r := 1 to n
    for each node u
        for each node v
            dist[u,v,r] := dist[u,v,r - 1]
            if dist[u,v,r] > dist[u,x_r, r - 1] + dist[x_r,v, r - 1]
                dist[u,v,r] := dist[u,x_r, r - 1] + dist[x_r,v, r - 1]
```

Time complexity: $O(n^3)$

Transitive Closure of a directed graph

- Given directed graph $G = (V, E)$ with vertex set $V = \{1,2,\ldots,n\}$, define the **transitive closure** of G as the graph $G^* = (V, E^*)$, where
  - $E^* = \{(i, j)$: there is a path from vertex i to j in $G\}$.
- Just assign weight 1 to each edge, and run Floyd-Warshall. Then if there is a path between u and v, $dist(u, v) < n$, otherwise $dist(u, v) = \infty$
- Or alternatively (and more efficiently), use $\vee$ (logical Or) and $\wedge$ (logical And) for the arithmetic operations $\min$ and $+$, and Define $t_{u,v}^{(k)}$ to indicate if there is a path from u to v with all intermediate vertices in $\{1,2,\ldots,k\}$:
  - $t_{uv}^{(0)} = \begin{cases} 0, & if\ u \neq v\ and\ (u, v) \notin E \\ 1, & if\ u = v\ or\ (u, v) \in E \end{cases}$
  - For $k \geq 1$, $t_{u,v}^{(k)} = t_{u,v}^{(k-1)} \vee \left(t_{u,x_k}^{(k-1)} \wedge t_{x_k,v}^{(k-1)} \right)$

Application of APSP: Compute Transitive Closure

FloydWarshallAPSP(G):

```
for each pair (u,v) in V*V
    if (u, v) in E then dist[u,v, 0] := w(u, v)
    else dist[u,v,0] := INF
for r := 1 to n
    for each node u
        for each node v
            dist[u,v,r] := dist[u,v,r - 1]
            if dist[u,v,r] > dist[u,x_r, r - 1] + dist[x_r,v, r - 1]
                dist[u,v,r] := dist[u,x_r, r - 1] + dist[x_r,v, r - 1]
```

FloydWarshallTransitiveClosure(G):

```
for each pair (u,v) in V*V
    if (u, v) in E then t[u,v, 0] := TRUE
    else t[u,v,0] := FALSE
for r := 1 to n
    for each node u
        for each node v
            t[u,v,r] := t[u,v,r - 1]
            if t[u,x_r, r - 1] AND t[x_r,v, r - 1]
                t[u,v,r] := TRUE
```

Further reading

- [CLRS] Ch.25
- [Erickson] Ch.9