智能软件与工程学院
School of Intelligent Software and Engineering

动态规划
Dynamic Programming

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2023 Fall

The slides are mainly adapted from the original ones shared by Chaodong Zheng and Kevin Wayne.Thanks for their supports!

Problem Solving Strategies

- Divide and Conquer
  - Divide (reduce) the problem into one or more subproblems;
  - Recursively solve subproblems;
  - Combine partial solutions to obtain complete solution.
  - Example: merge-sort, quick-sort, binary-search, …

- Greedy
  - Gradually generate a solution for the problem;
  - At each step: make an greedy choice, then compute optimal solution of the subproblem induced by the choice made.
  - Example: MST, Dijkstra, Huffman codes, …

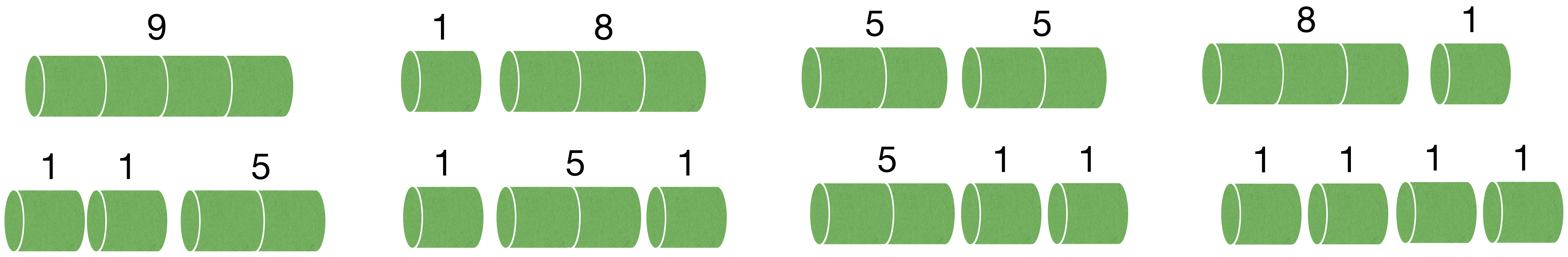
What if a problem does not exhibit greedy choice property?

The Rod-Cutting Problem

- Assume we are given a rod of length n. We sell length i rod for a price of p_i, where $i \in \mathbb{N}^+$ and $1 \leq i \leq n$.

i	1	2	3	4	5	6	7	8	9	10
p_i	1	5	8	9	10	17	17	20	24	30

- How to cut the rod to gain maximum revenue?
- Enumerate all possibilities?
  - There are 2^{n-1} ways to cut up a length n rod…



8 possible ways of cutting up a rod of length 4 and their prices

The Rod-Cutting Problem

- Greedy algorithm?
- Let r_k denote max profit for a length k rod.
- Optimal substructure property:
    - $r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$
- Greedy choice property?
    - Always cut at the most profitable position? ($\max(\frac{p_i}{i})$)
        - Unfortunately, it does **NOT** yield optimal solution! ($n = 3, p_1 = 1, p_2 = 7, p_3 = 9$)

The Rod-Cutting Problem

- Let r_k denote max profit for a length k rod.
- Optimal substructure property holds.
  $(r_n = \max_{1 \le i \le n} (p_i + r_{n-i}))$
- Optimal substructure property already implies an algorithm! (even though without greedy choice property)
  - At each step, enumerate all possible cut.
  - For each cut, (recursively) find optimal solution. (Find all r_{n-i})
  - Find optimal solution for original problem. (Find $\max_{1 \le i \le n} (p_i + r_{n-i})$)

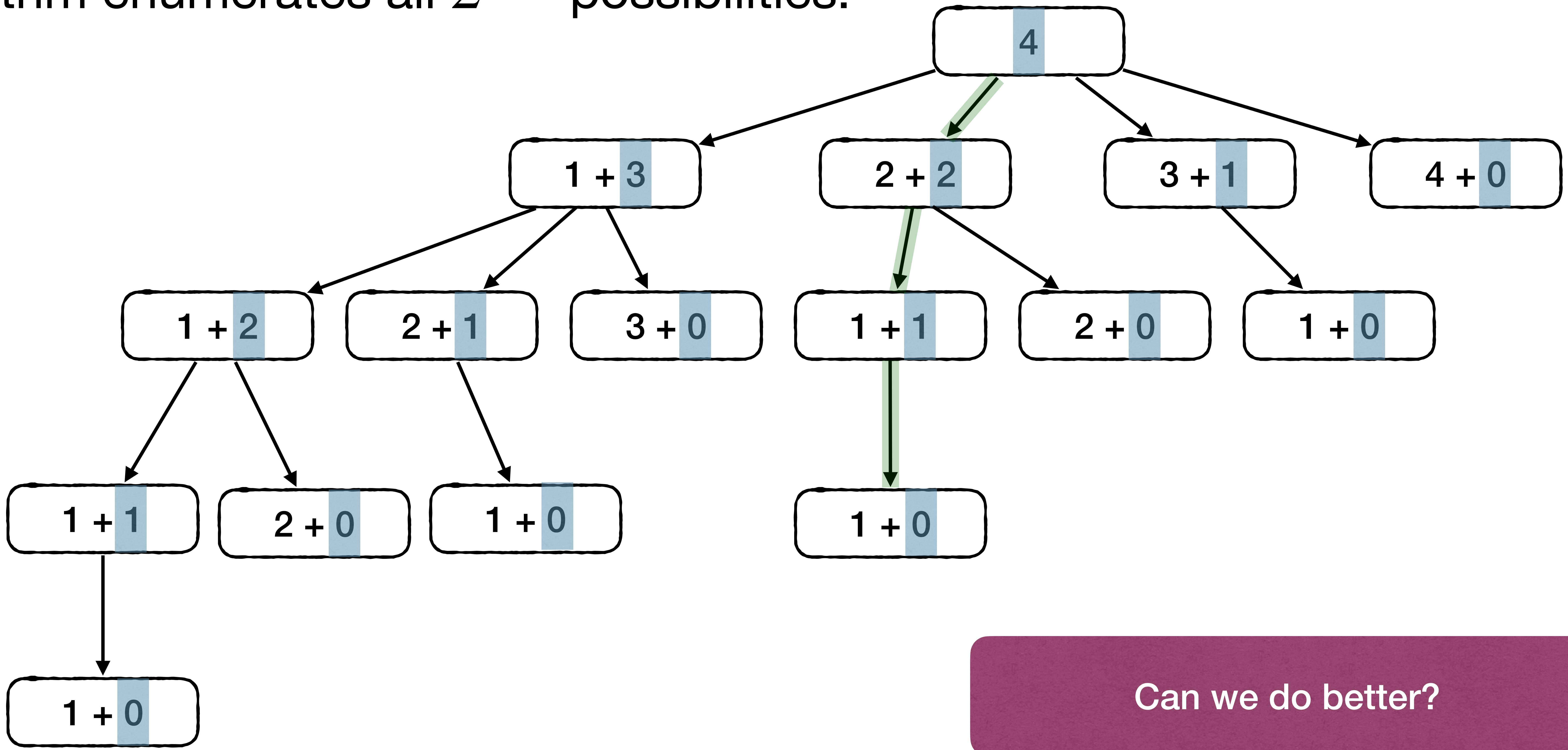
A simple recursive algorithm

```
CutRodRec(prices,n):
if n = 0
    return 0
r := -INF
for i := 1 to n
    r := Max(r, prices[i] + CutRodRec(prices,n-i) )
return r
```

Performance of this algorithm?

The Rod-Cutting Problem

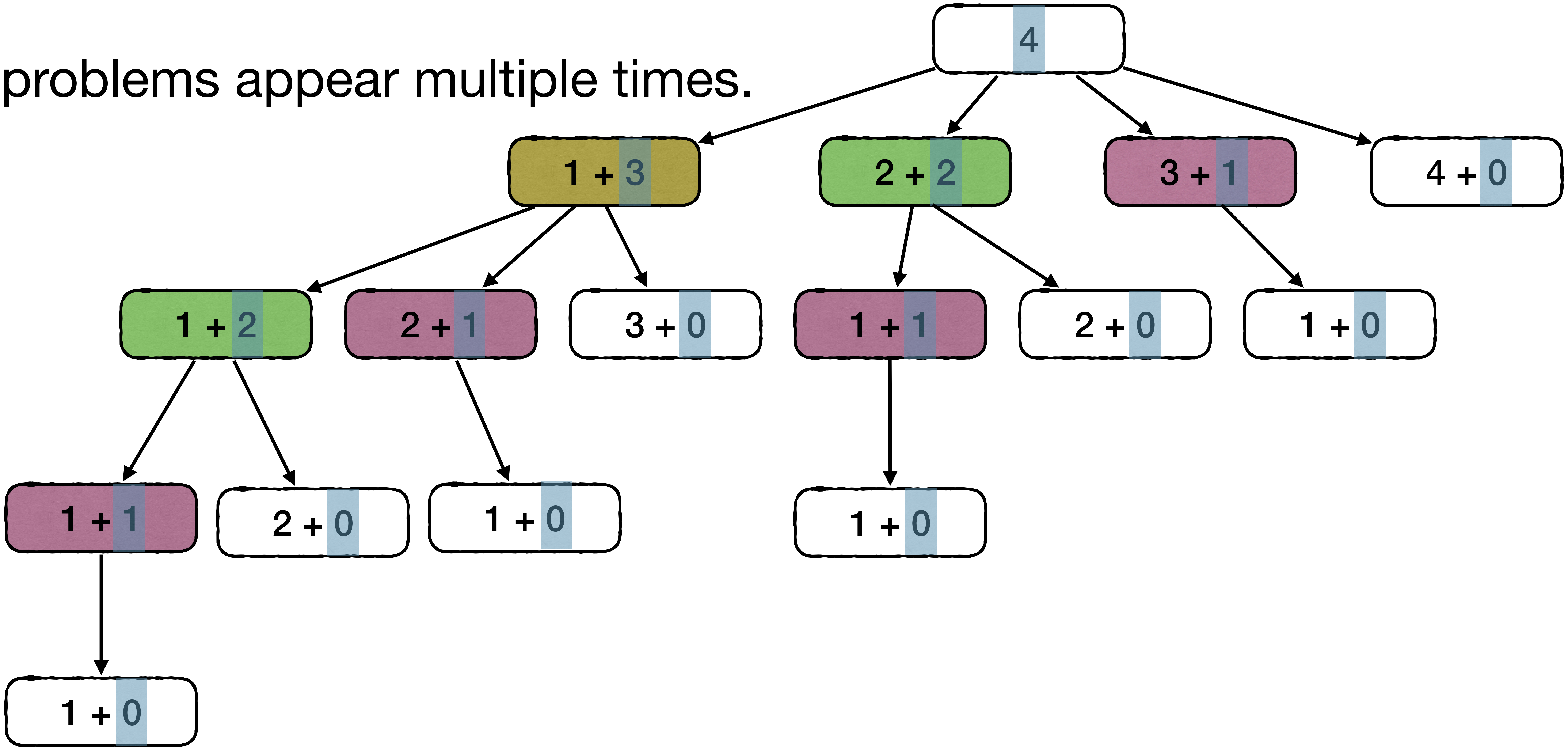
- Each path from root to a leaf denotes a way to cut the rod.
- This algorithm enumerates all 2^{n-1} possibilities!



Can we do better?

The Rod-Cutting Problem

- For each subproblem, only need to solve it once!
- Each node denotes a subproblem of certain size
- Some subproblems appear multiple times.



The Rod-Cutting Problem

- Solve each subproblem once and remember solution!

```
CutRodRecMem(prices,n):
for i := 0 to n
    r[i] := -INF
return CutRodRecMemAux(prices, r, n)
```

```
CutRodRecMemAux(prices,r,n):
if r[n] > 0
    return r[n]
if n = 0
    q := 0
else
    q := -INF
    for i := 1 to n
        q := Max(q, prices[i] + CutRodRecMemAux(prices, r, n-i) )
r[n] := q
return q
```

The Rod-Cutting Problem

- Runtime of this algorithm:
  - Each subproblem (optimal revenue for length i rod) is solved once.
  - When actually solving the size i problem, optimal solutions of subproblems are known. (Otherwise we would recurse first.)
    - Thus solving size i problem itself (without subproblems) needs $\Theta(i)$ time.
  - Total runtime is $\Theta(1 + 2 + \ldots + n) = \Theta(n^2)$.

The Rod-Cutting Problem

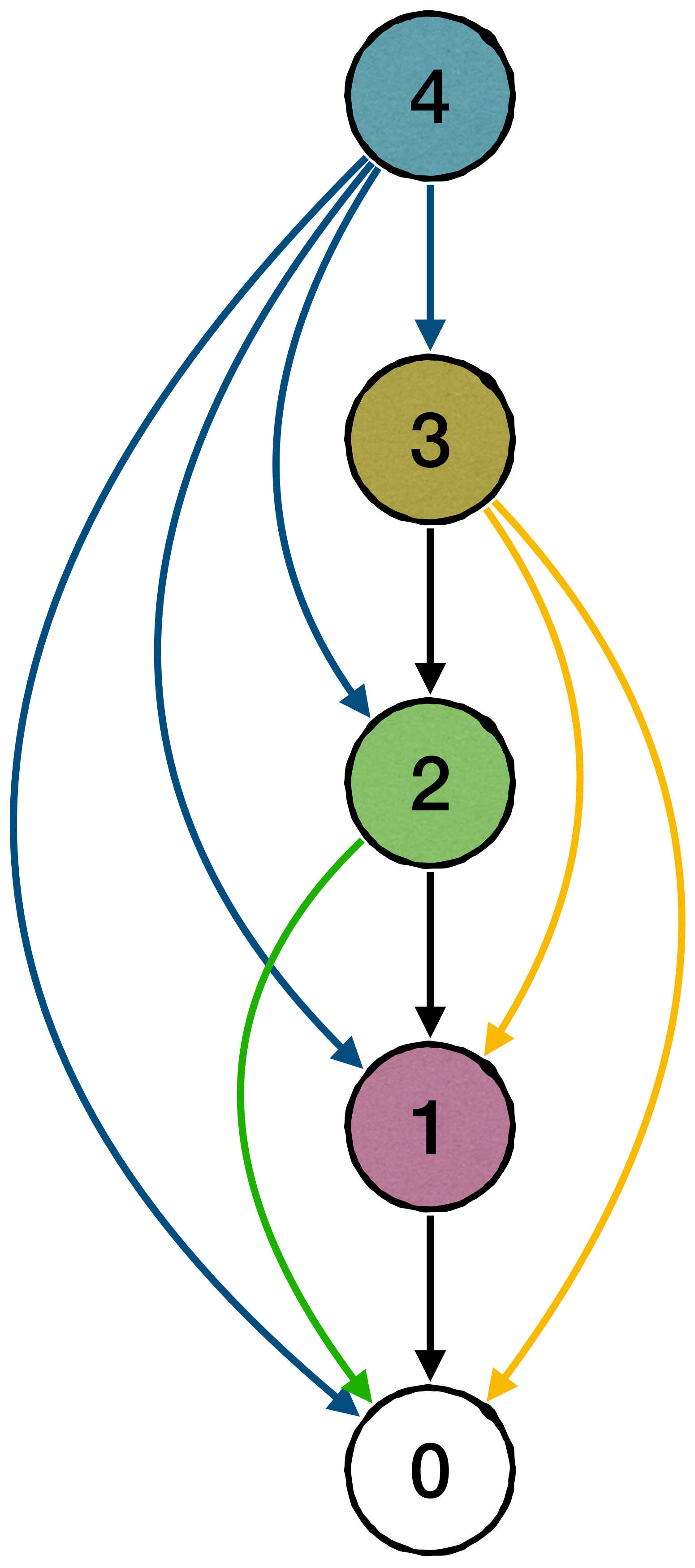
The Top-Down Approach

- Solving the problem using recursion is like DFS.
- Convert recursion to iteration?
  - A problem cannot be solved until all subproblems it depends upon are solved.
  - The subproblem graph is a DAG! (WHY?)
  - Consider subproblems in reverse topological order!

CutRodIter(prices,n):

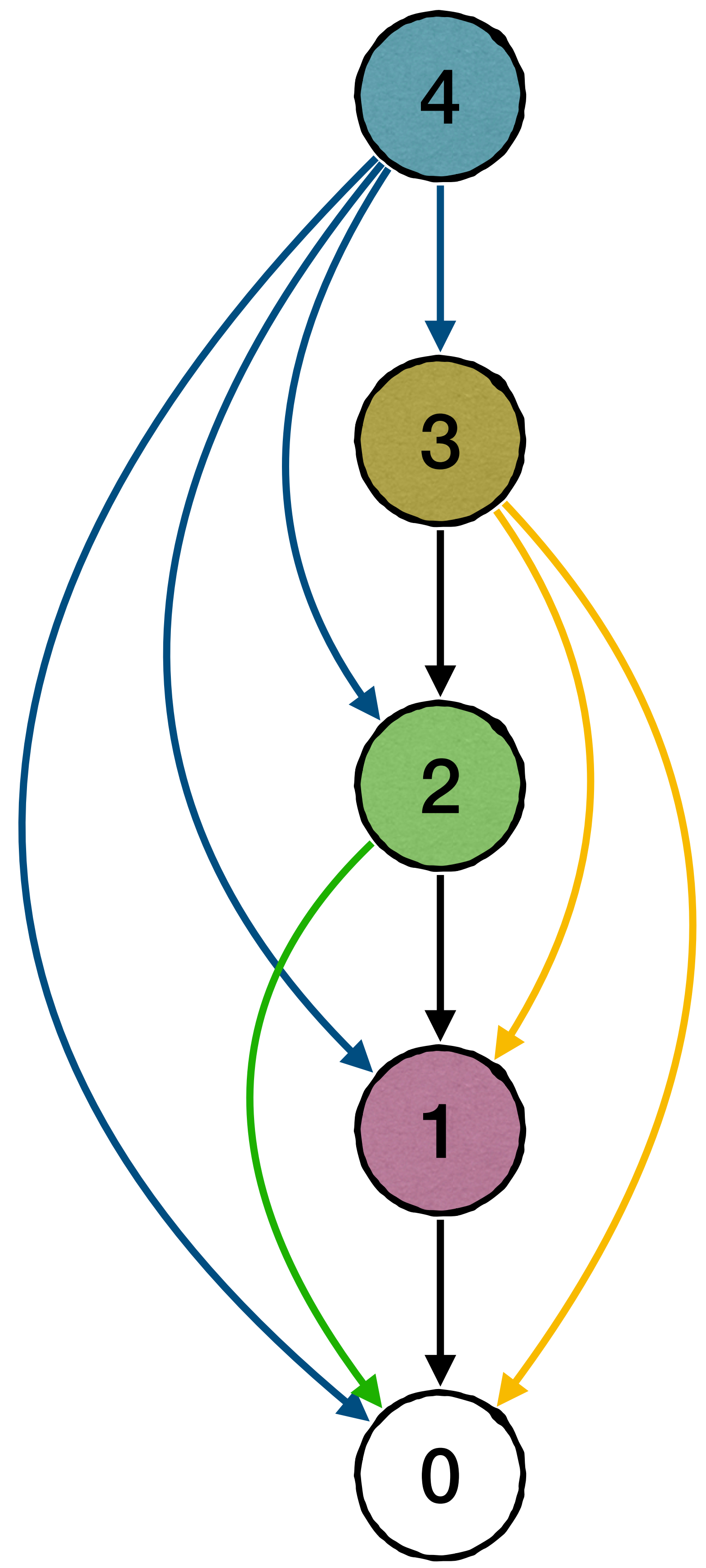
```
r[0] := 0
for i := 1 to n
    q := -INF
    for j := 1 to i
        q := Max(q, prices[j] + r[i - j])
    r[i] := q
return r[n]
```

Runtime is $\Theta(n^2)$



Reconstructing optimal solution

- Algorithm gives optimal revenue, but how to cut?



CutRodIter(prices,n):

```
r[0] := 0
for i := 1 to n
    q := -INF
    for j := 1 to i
        if q < prices[j] + r[i - j]
            q := prices[j] + r[i - j]
            cuts[i] = j
    r[i] := q
return r[n]
```

PrintOpt(cuts,n):

```
while n > 0
    Print cuts[n]
    n := n – cuts[n]
```

Dynamic Programming

Dynamic Programming (DP)

- Consider an (optimization) problem:
  - Build optimal solution step by step.
  - Problem has **optimal substructure** property.
    - We can design a recursive algorithm.
  - Problem has lots of **overlapping** subproblems.
    - Recursion and memorize solutions. (Top-Down)
    - Or, consider subproblems in the right order. (Bottom-Up)
- We have seen such algorithms previously!

The Floyd-Warshall Algorithm

- **Strategy:** recuse on the *set of node* the shortest paths use.

- Define $dist(u, v, r)$ be length of shortest path from u to v, s.t. only nodes in $V_r = \{x_1, x_2, ..., x_r\}$ can be ***intermediate*** nodes in paths.

- $$dist(u, v, r) = \begin{cases} w(u, v) & if\ r = 0\ and\ (u, v) \in E \\ \infty & if\ r = 0\ and\ (u, v) \notin E \\ \min \begin{Bmatrix} dist(u, v, r-1) \\ dist(u, x_r, r-1) + dist(x_r, v, r-1) \end{Bmatrix} & otherwise \end{cases}$$

The Floyd-Warshall Algorithm

```
FloydWarshallAPSP(G):
for each pair (u,v) in V*V
    if (u, v) in E  then dist[u,v, 0] := w(u, v)
    else dist[u,v,0] := INF
for r := 1 to n
    for each node u
        for each node v
            dist[u,v,r] := dist[u,v,r - 1]
            if dist[u,v,r] > dist[u,x_r, r - 1] + dist[x_r,v, r - 1]
                dist[u,v,r] := dist[u,x_r, r - 1] + dist[x_r,v, r - 1]
```

Bottom-up Approach

Developing a DP algorithm

- Characterize the structure of solution.
  - E.g. [rod-cutting]: (one cut of length i) + (solution for length $n-i$)
- Recursively define the value of an optimal solution.
  - E.g. [rod-cutting]: $r_n = \max_{1 \le i \le n} (p_i + r_{n-i})$
- Compute the value of an optimal solution.
  - Top-down or Bottom-up. (Usually use bottom-up)
- [$*$] Construct an optimal solution.
  - Remember optimal choices (beside optimal solution values).

Matrix-chain Multiplication

- Input: Matrices $A_1, A_2, \ldots, A_n$, with A_i of size $p_{i-1} \times p_i$.
- Output: $A_1 A_2 \ldots A_n$.
- Problem: Compute output with minimum work?
- Matrix multiplication is associative, and order does matter!
  - Example: $|A_1| = 10 \times 100, |A_2| = 100 \times 5, |A_3| = 5 \times 50$
  - $(A_1A_2)A_3$ costs $10 \times 100 \times 5 + 10 \times 5 \times 50 = 7500$
  - $A_1(A_2A_3)$ costs $100 \times 5 \times 50 + 10 \times 100 \times 50 = 75000$

Optimal order for minimum cost?

Developing a DP algorithm for Matrix-chain Multiplication

- Characterize the structure of solution.
  - What's the last step in computing $A_1A_2...A_n$?
  - For every order, last step is $(A_1A_2...A_k) \cdot (A_{k+1}A_{k+2}...A_n)$.
  - In general, $A_iA_{i+1}...A_j = (A_iA_{i+1}...A_k) \cdot (A_{k+1}A_{k+2}...A_j)$
- Recursively define the value of an optimal solution.
  - Let $m[i,j]$ be the minimal cost for computing $A_iA_{i+1}...A_j$
  - $m[i,j] = \min_{i \le k < j} (m[i,k] + m[k+1,j] + p_{i-1}p_kp_j)$
    - Optimal Substructure Property!

Developing a DP algorithm for Matrix-chain Multiplication

- Let $m[i,j]$ be the minimal cost for computing $A_iA_{i+1}\ldots A_j$
- $m[i,j] = \min_{i \le k < j} (m[i,k] + m[k+1,j] + p_{i-1}p_kp_j)$
- Compute the value of an optimal solution.
  - Top-down (recursion with memorization) is easy, but bottom-up?
  - What does $m[i,j]$ depend upon?
    - $m[i,j]$ depend upon $m[i',j']$, where $j' - i' < j - i$.
  - Compute $m[i,j]$ in length increasing order!

```
MatrixChainDP(A1, A2,...,An):
for i := 1 to n
    m[i, i] := 0
for l := 2 to n
    for i := 1 to n - l +1
        j := i + l -1
        m[i, j] = INF
        for k := i to j - 1
            cost := m[i,k] + m[k+1,j] + p_{i-1}*p_k*p_j
            if cost < m[i, j]
                m[i, j] := cost
return m
```

Developing a DP algorithm for Matrix-chain Multiplication

- Construct an optimal solution.
  - For each (i, j) pair, remember the position of the optimal "split".

MatrixChainDP($A_1, A_2, \ldots, A_n$):

```
for i := 1 to n
    m[i, i] := 0
for l := 2 to n
    for i := 1 to n - l +1
        j := i + l -1
        m[i, j] = INF
        for k := i to j - 1
            cost := m[i,k] + m[k+1,j] + p_{i-1}*p_k*p_j
            if cost < m[i, j]
                m[i, j] := cost
                s[i, j] := k
return <m, s>
```

MatrixChainPrintOpt(s,i,j):

```
if i = j
    Print "A_i"
else
    Print "("
    MatrixChainPrintOpt(s, i, s[i,j])
    MatrixChainPrintOpt(s, s[i,j]+1, j)
    Print ")"
```

Edit Distance

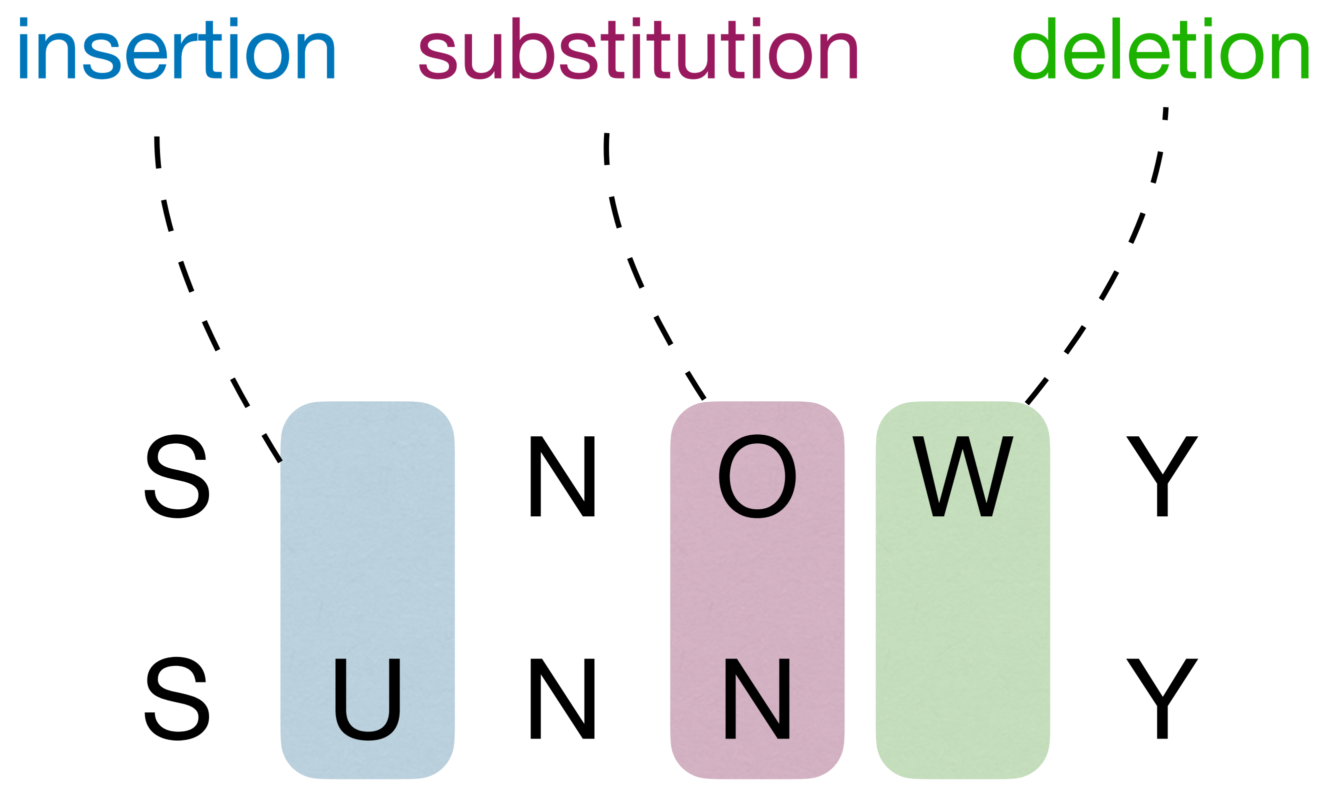
- Given two strings, how *similar* are they?
  - **Application:** when a spell checker encounters a possible misspelling, it needs to search dictionary to find *nearby* words.
- Consider following three type of operations for a string:
  - **Insertion:** insert a character at a position.
  - **Deletion:** remove a character at a position.
  - **Substitution:** change a character to another character.

Edit Distance

- Edit Distance of A and B: minimal number of ops to transform A into B.
- *Example*: transform "`SNOWY`" to "`SUNNY`"
  - Insertion: `SNOWY -> SUNOWY`
  - Deletion: `SUNOWY -> SUNOY`
  - Substitution: `SUNOY -> SUNNY`
  - Edit distance is *at most* 3 (and it *indeed* is 3).

Edit Distance

- Edit Distance of A and B: minimal number of ops to transform A into B.
  - Operations: **Insertion**, **Deletion**, and **Substitution**.
- One way to visualize the editing process:
  - **Align** string A above string B;
  - A gap in first line indicates an insertion (to A);
  - A gap in second line indicates a deletion (from A);
  - A column with different characters indicates a substitution.



Edit Distance

- **Problem:** Given A and B, what is the edit distance?
- **Step 1**: Characterize the structure of solution.
  - Consider transform A[1 … m] to B[1 … n].
  - Each solution can be visualized in the way described earlier.
  - Last column must be one of three cases: $\begin{matrix}-\\ B[n]\end{matrix}$ or $\begin{matrix}A[m]\\ B[n]\end{matrix}$ or $\begin{matrix}A[m]\\ -\end{matrix}$
  - Each case reduces the problem to a subproblem:
    - $(-, B[n])$: edit distance of A[1 … m] and B[1 … (n - 1)]
    - $(A[m], B[n])$: edit distance of A[1 … (m - 1)] and B[1 … (n - 1)]
    - $(A[m], -)$: edit distance of A[1 … (m - 1)] and B[1 … n]

S		N	O	W	Y
S	U	N	N		Y

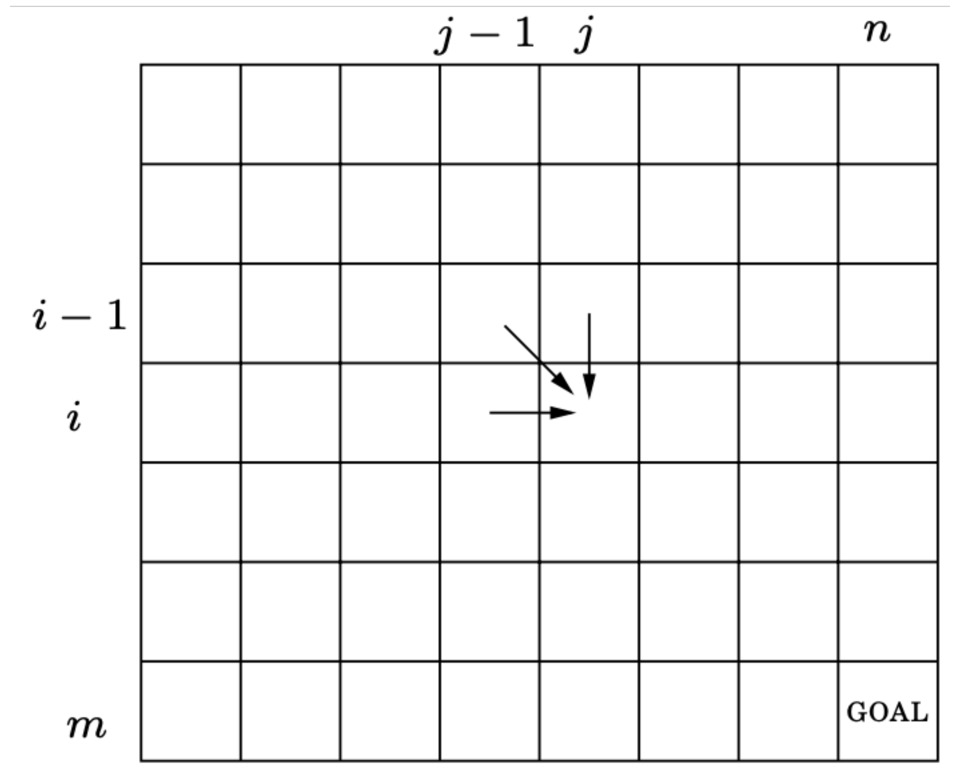
Edit Distance

- Step 2: Recursively define the value of an optimal solution

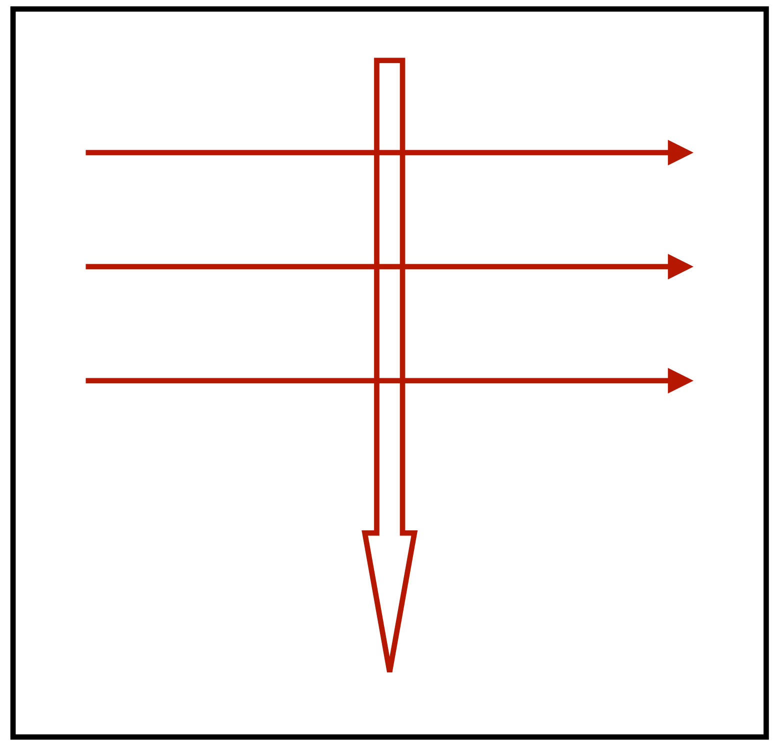
$$dist(i,j) = \begin{cases} i & \text{if } j = 0 \\ j & \text{if } i = 0 \\ \min \left\{ \begin{array}{l} dist(i,j-1)+1 \\ dist(i-1,j)+1 \\ dist(i-1,j-1)+I[A[i]=B[j]] \end{array} \right\} & \text{otherwise} \end{cases}$$

Edit Distance

- Step 3: Compute the value of an optimal solution (Bottom-Up).
  - What does $dist(i, j)$ depend upon?



  - Outer-loop:
    - increasing i;
  - Inner-loop:
    - increasing j

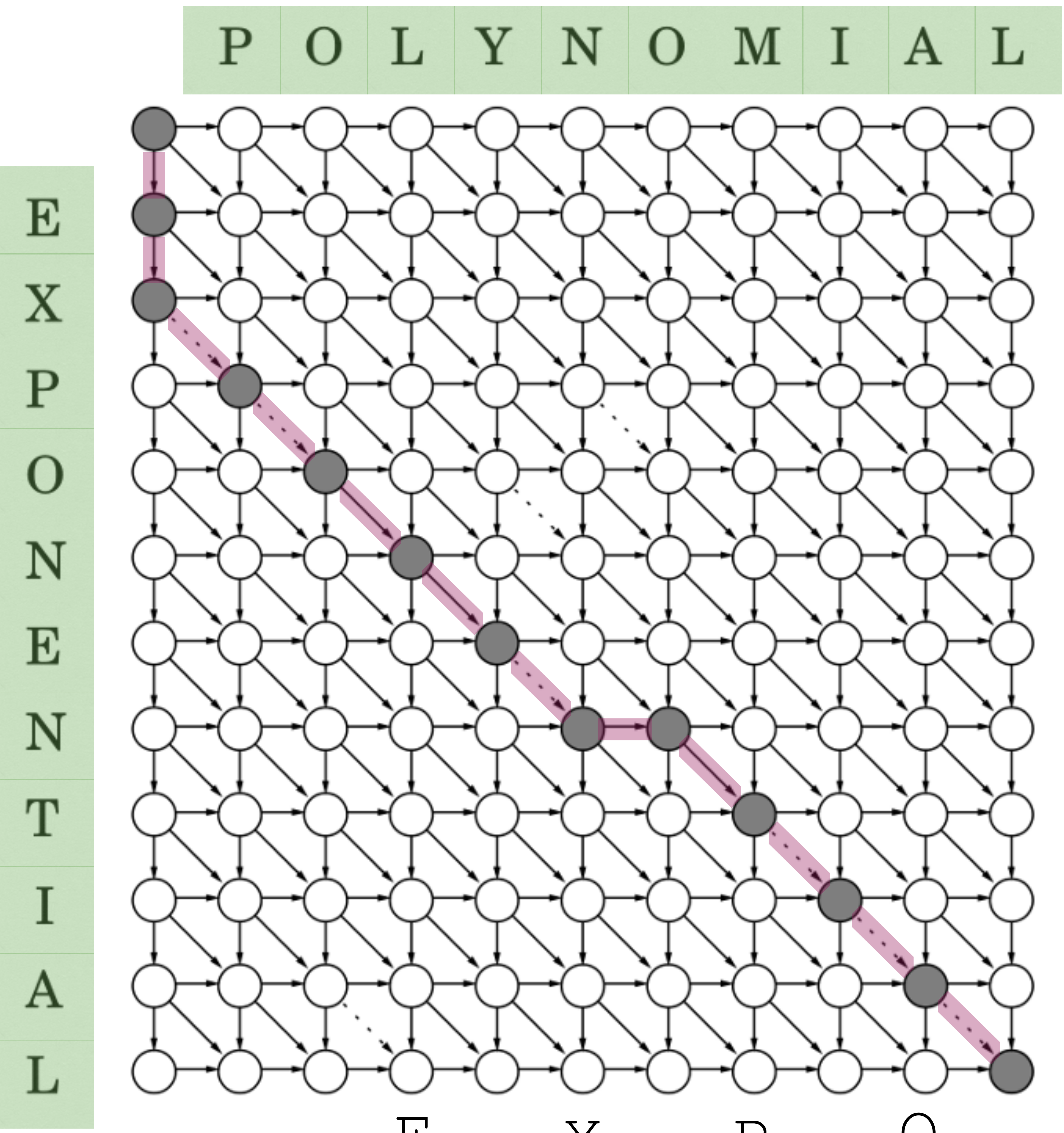


<u>EditDistDP(A[1…m],B[1…n]):</u>

for $i := 0$ **to** m
    $dist[i, 0] := i$
for $j := 0$ **to** n
    $dist[0, j] := j$
for $i := 1$ **to** m
    for $j := 1$ **to** n
        $delDist := dist[i - 1, j] + 1$
        $insDist := dist[i, j - 1] + 1$
        $subDist := dist[i - 1, j - 1] + Diff(A[i], B[j])$
        $dist[i, j] := Min(delDist, insDist, subDist)$
return $dist$

Step 4: Construct an optimal solution.

Subproblem graph



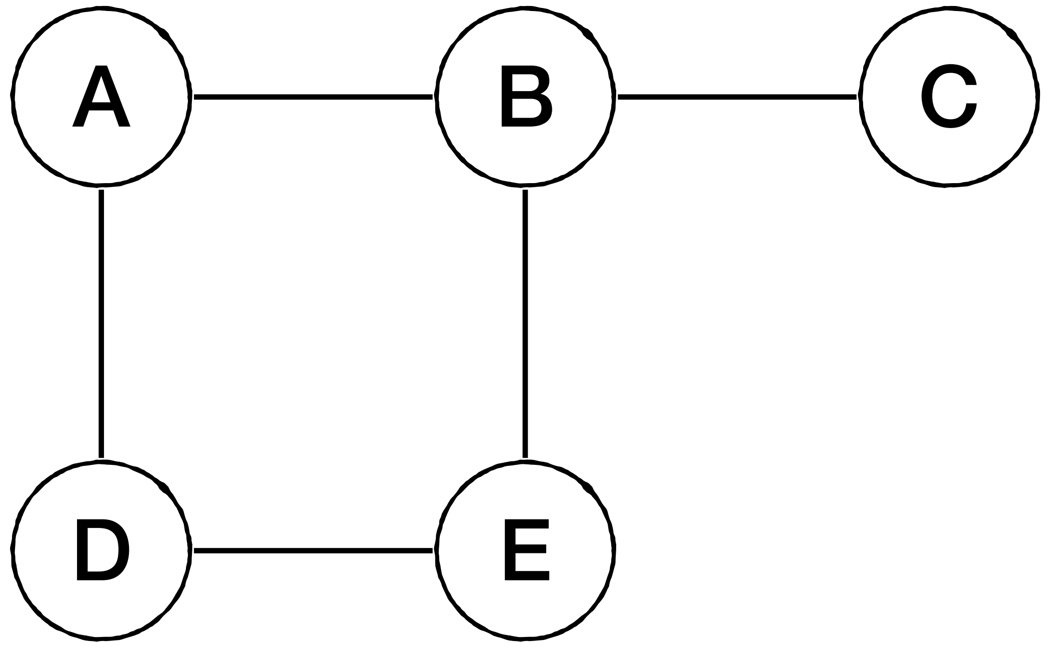
- DAG Transform “EXPONENTIAL” to “POLYNOMIAL”
  - → Insertion (costs 1)
  - ↓ Deletion (costs 1)
  - ↘ Substitution [diff] (costs 1)
  - ⇣↘ Substitution (costs 0)
- Edit distance:
  - Shortest path from top-left to right-bottom.

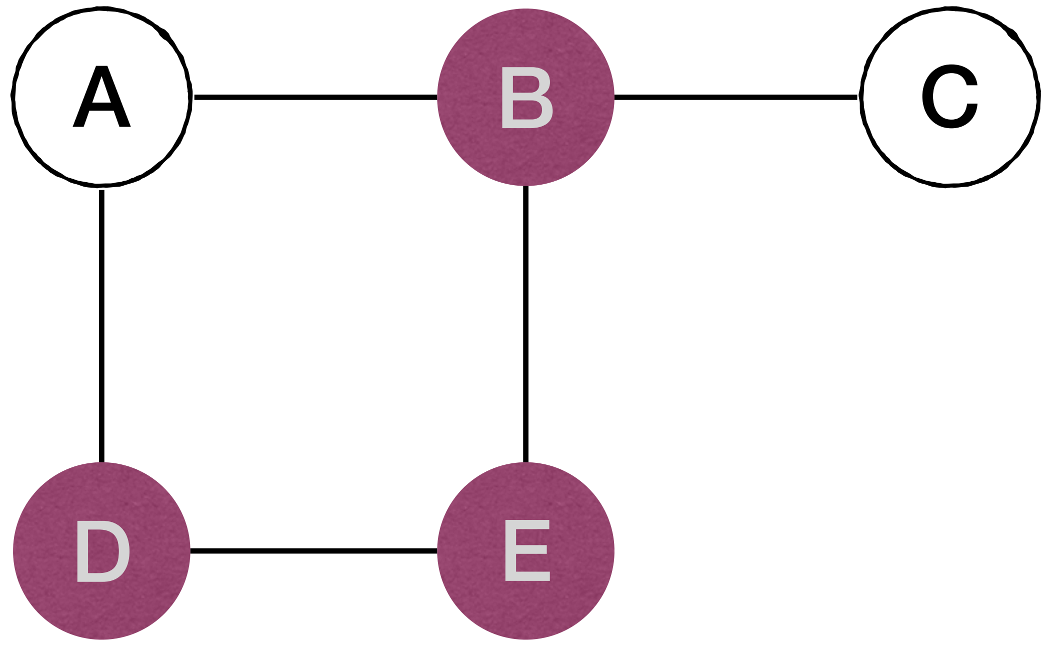
E	X	P	O	N	E	N	—	T	I	A	L
—	—	P	O	L	Y	N	O	M	I	A	L
1	1			1	1		1	1			

Maximum Independent Set

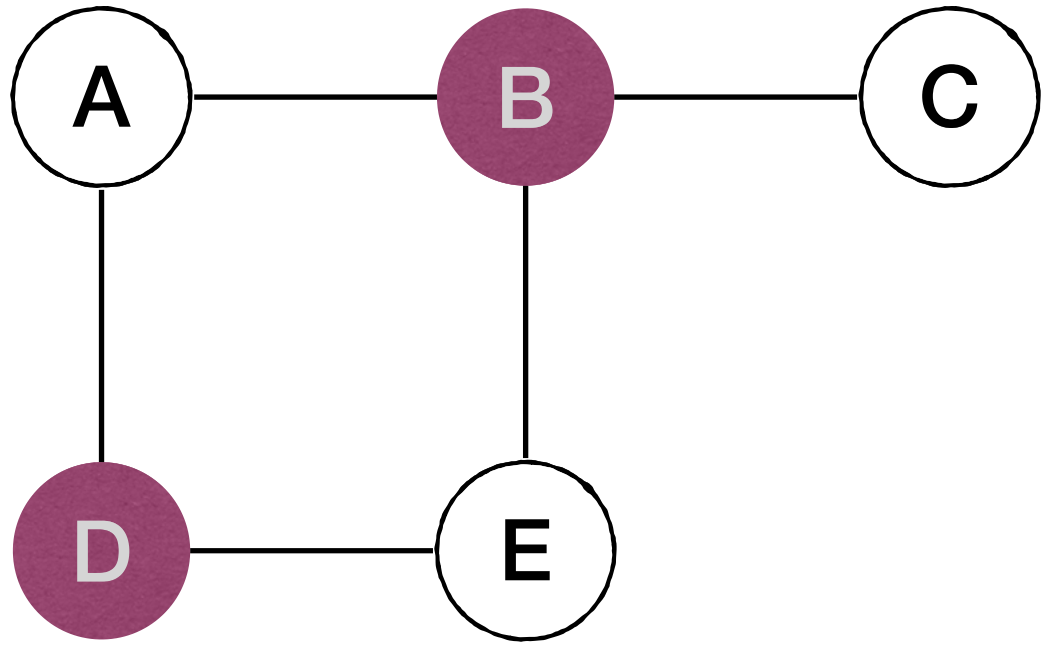
- Given an undirected graph $G = (V, E)$, an **independent set** I is a subset of V, such that no vertices in I are adjacent. Put another way, for all $(u, v) \in I \times I$, we have $(u, v) \notin E$.

- A **maximum independent set** (**MaxIS**) is an independent set of maximum size.

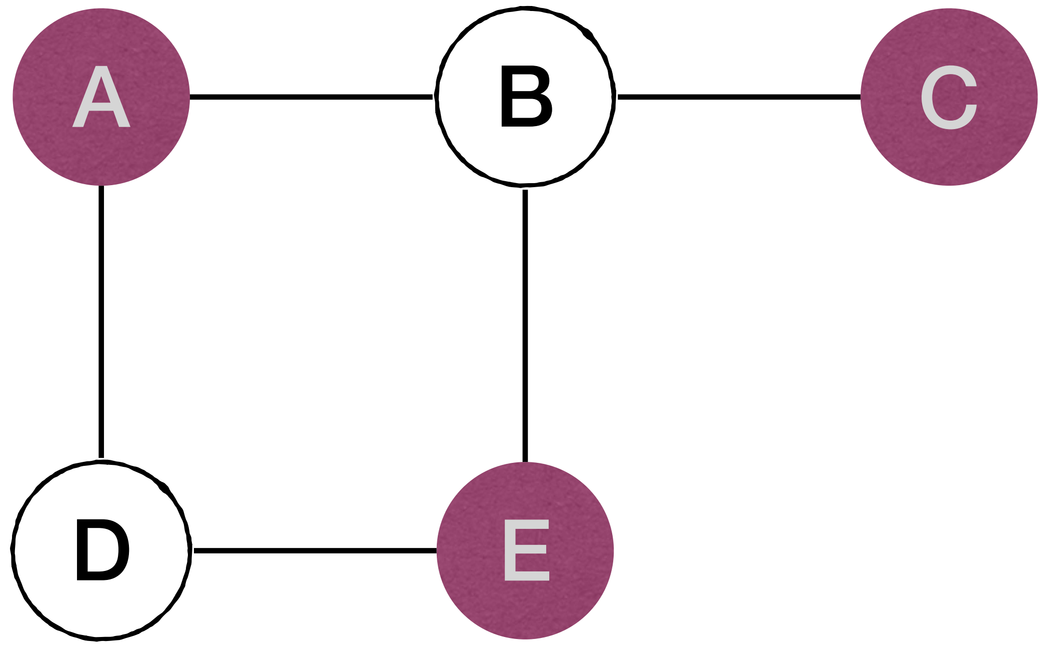




$\{B, D, E\}$ is Not IS



$\{B, D\}$ is IS,
but is Not MaxIS



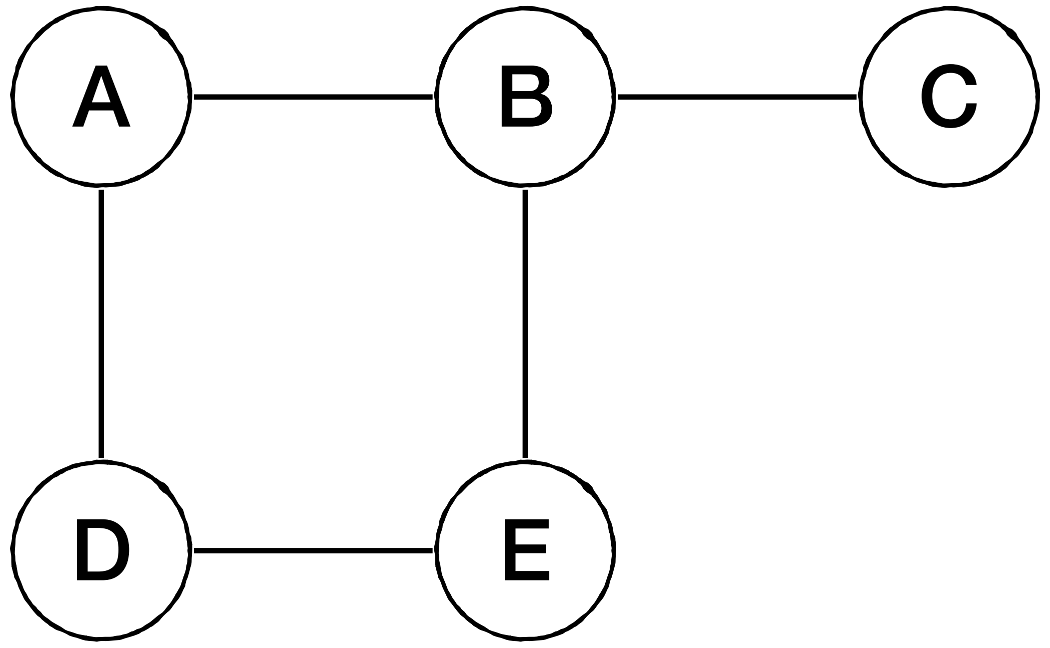
$\{A, E, C\}$ is IS,
and is also MaxIS

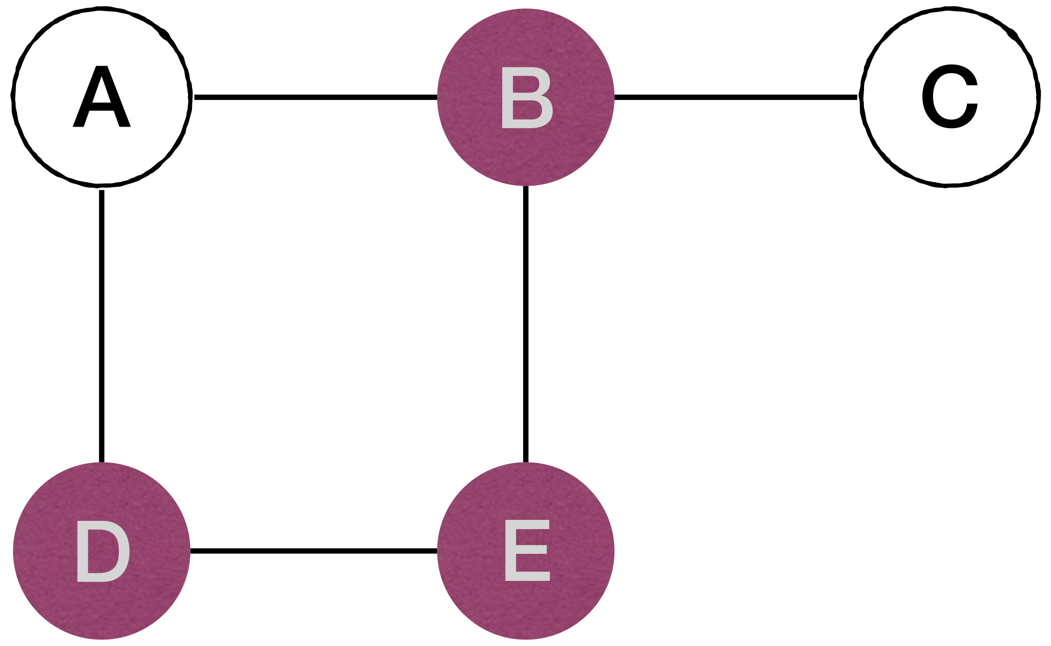
Maximum Independent Set

NP-hard!

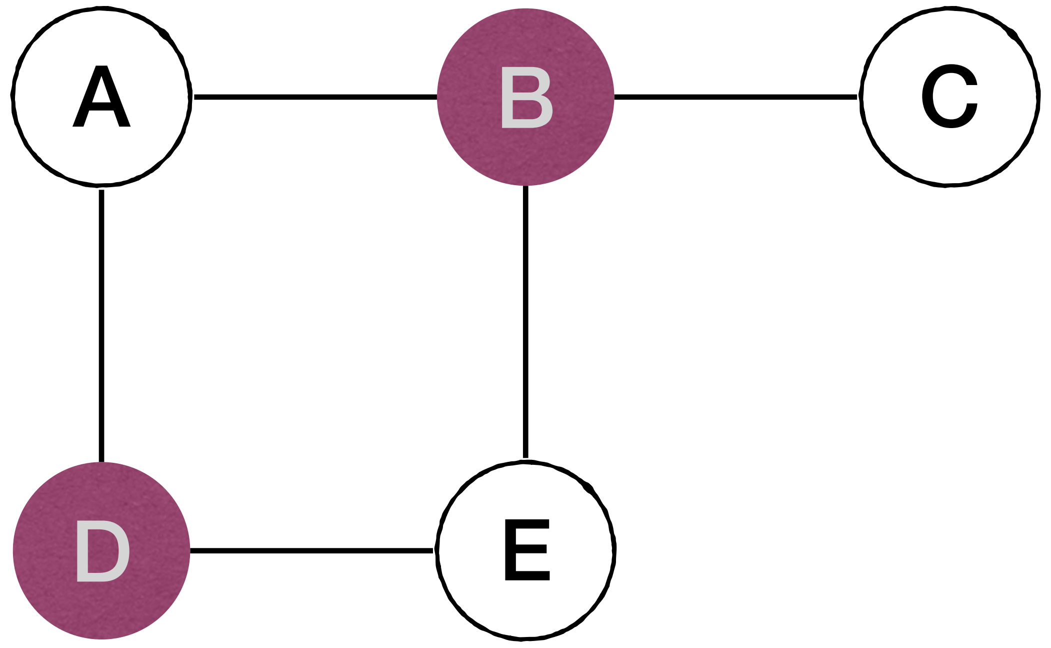
- Computing MaxIS in an arbitrary graph is very hard. Even getting an **approximate** MaxIS is very hard!

- But if we only consider **trees**, MaxIS is very easy!

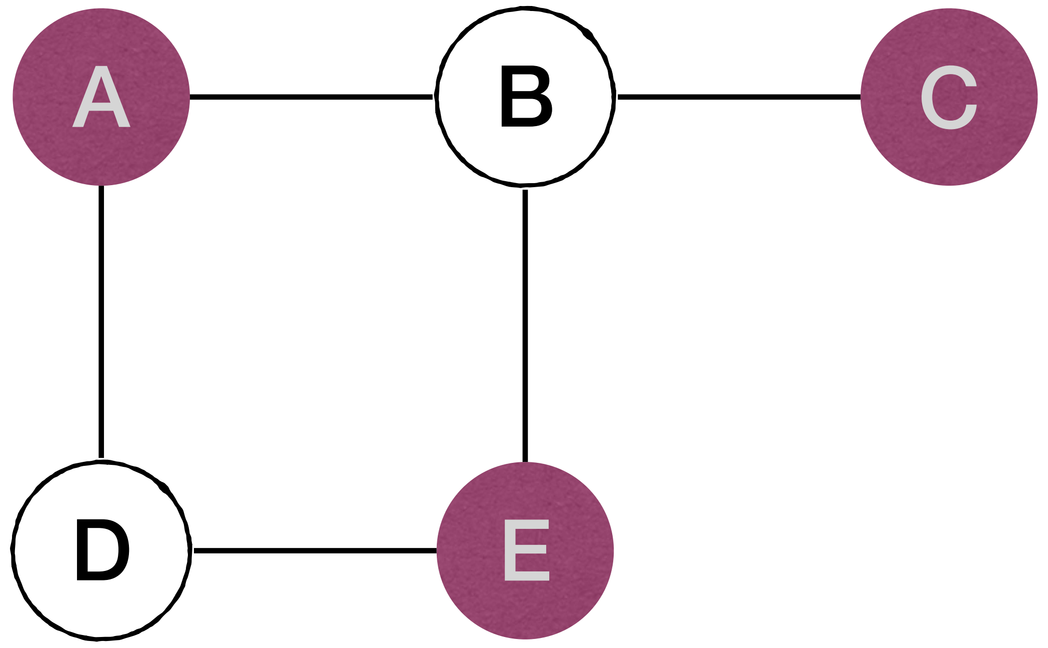




$\{B, D, E\}$ is Not IS



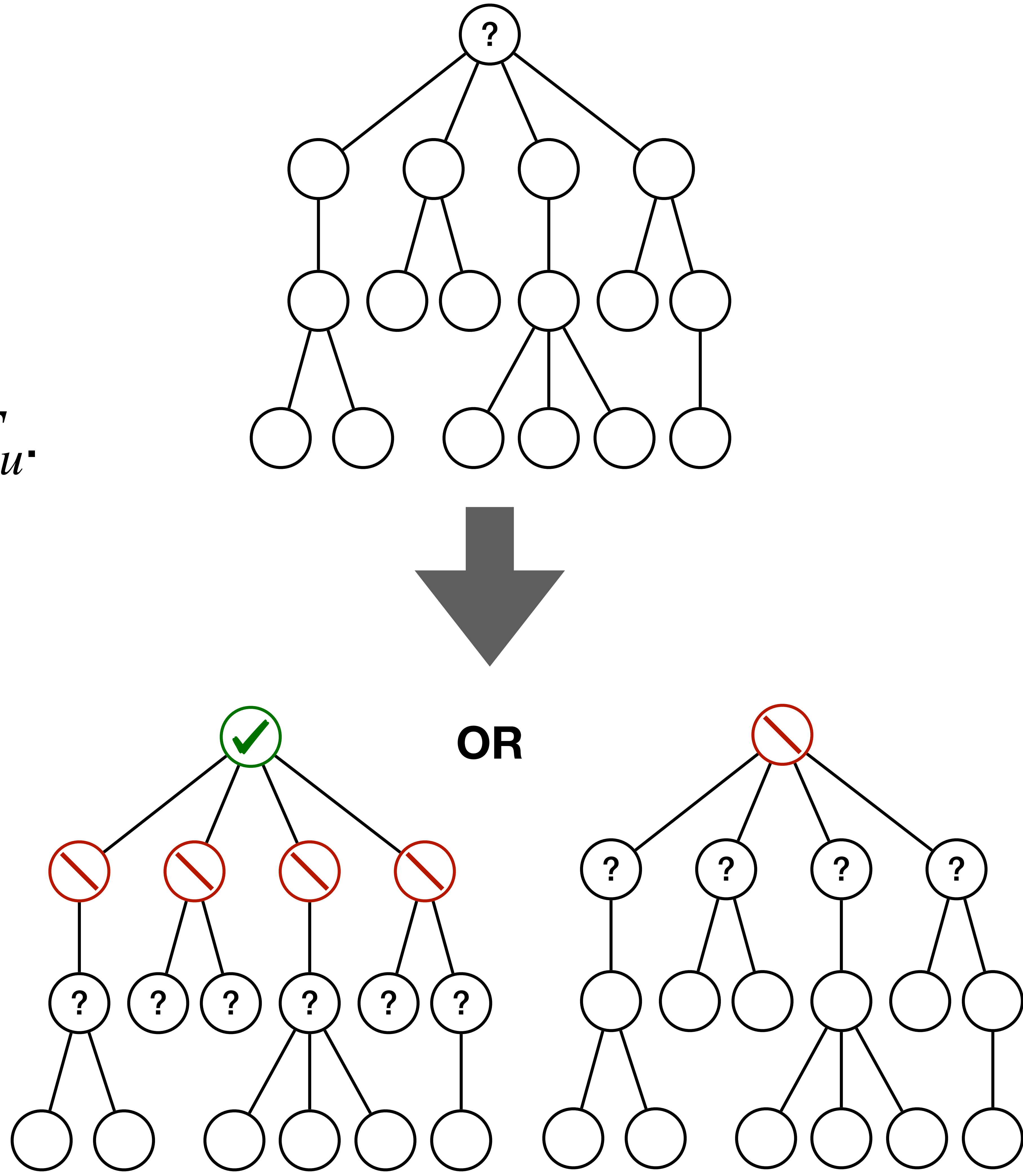
$\{B, D\}$ is IS,
but is Not MaxIS



$\{A, E, C\}$ is IS,
and is also MaxIS

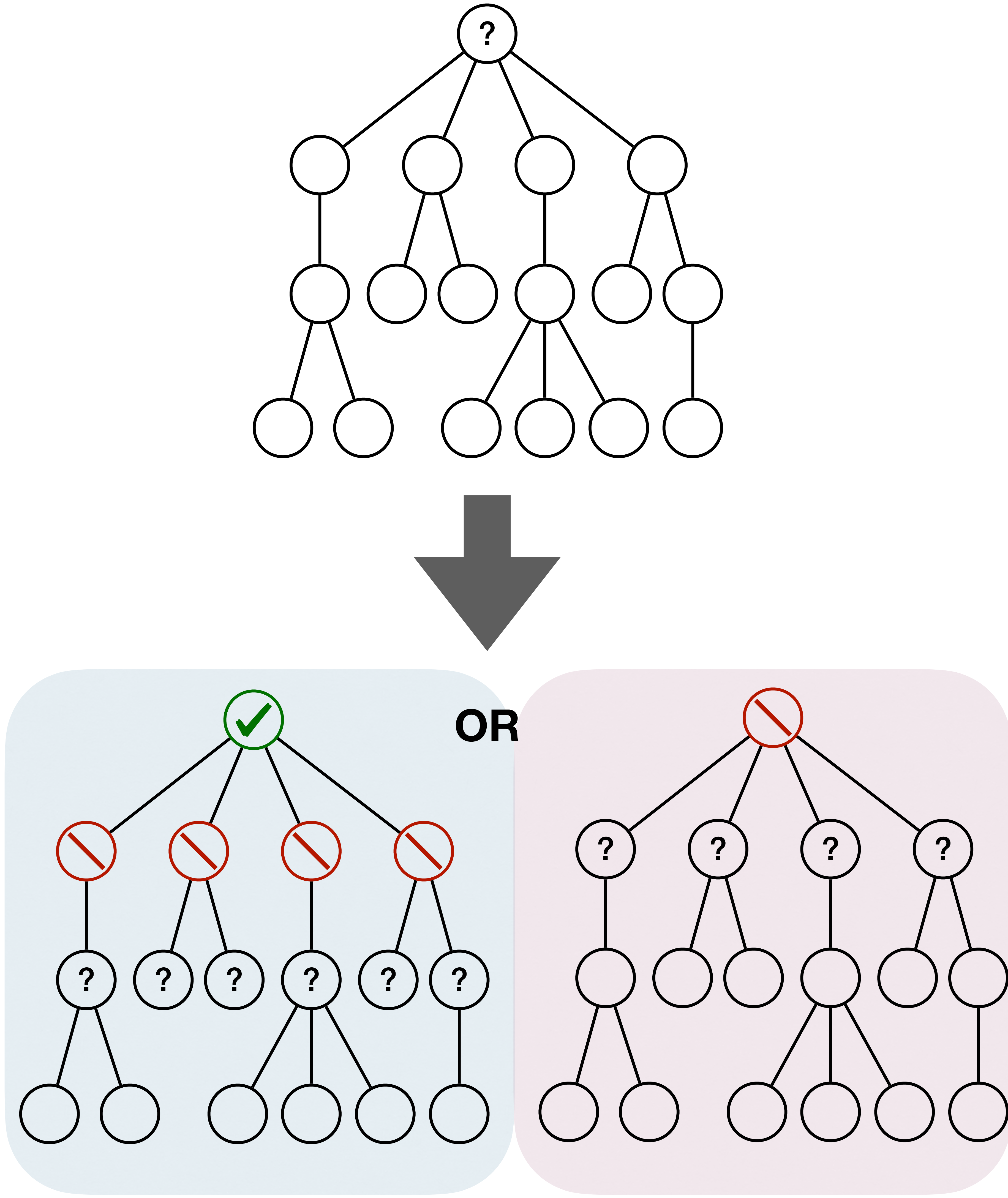
MaxIS of Trees

- **Problem:** Given a tree T with root r, compute a MaxIS of it.
- Step 1: Characterize the structure of solution.
  - Given an IS I of T, for each child u of r, set $I \cap V(T_u)$ is an IS of T_u.
- Step 2: Recursively define the value of an optimal solution.
  - Let $mis(T_u)$ be size of MaxIS of (sub)tree rooted at node u.
  - $$mis(T_u) = 1 + \sum_{v \text{ is a child of } u} mis(T_v)$$
  - **NO!** The recurrence depends on whether u in the MaxIS of T_u.



MaxIS of Trees

- Step 2: Recursively define the value of an optimal solution
  - Let $mis(T_u)$ be size of MaxIS of (sub)tree rooted at node u.
  - The recurrence depends on whether u in the MaxIS of T_u.
  - Let $mis(T_u,1)$ be size of MaxIS of T_u, s.t. u in the MaxIS.
  - Let $mis(T_u,0)$ be size of MaxIS of T_u, s.t. u NOT in the MaxIS.
  - $$mis(T_u,1) = 1 + \sum_{v \text{ is a child of } u} mis(T_v,0)$$
  - $$mis(T_u,0) = \sum_{v \text{ is a child of } u} mis(T_v)$$
  - $mis(T_u) = \max\{mis(T_u,0), mis(T_u,1)\}$



MaxIS of Trees

- Step 3: Compute the value of an optimal solution.

MaxIsDP(u):

$mis1 := 1$

$mis0 := 0$

for each $child\ v$ **of** u

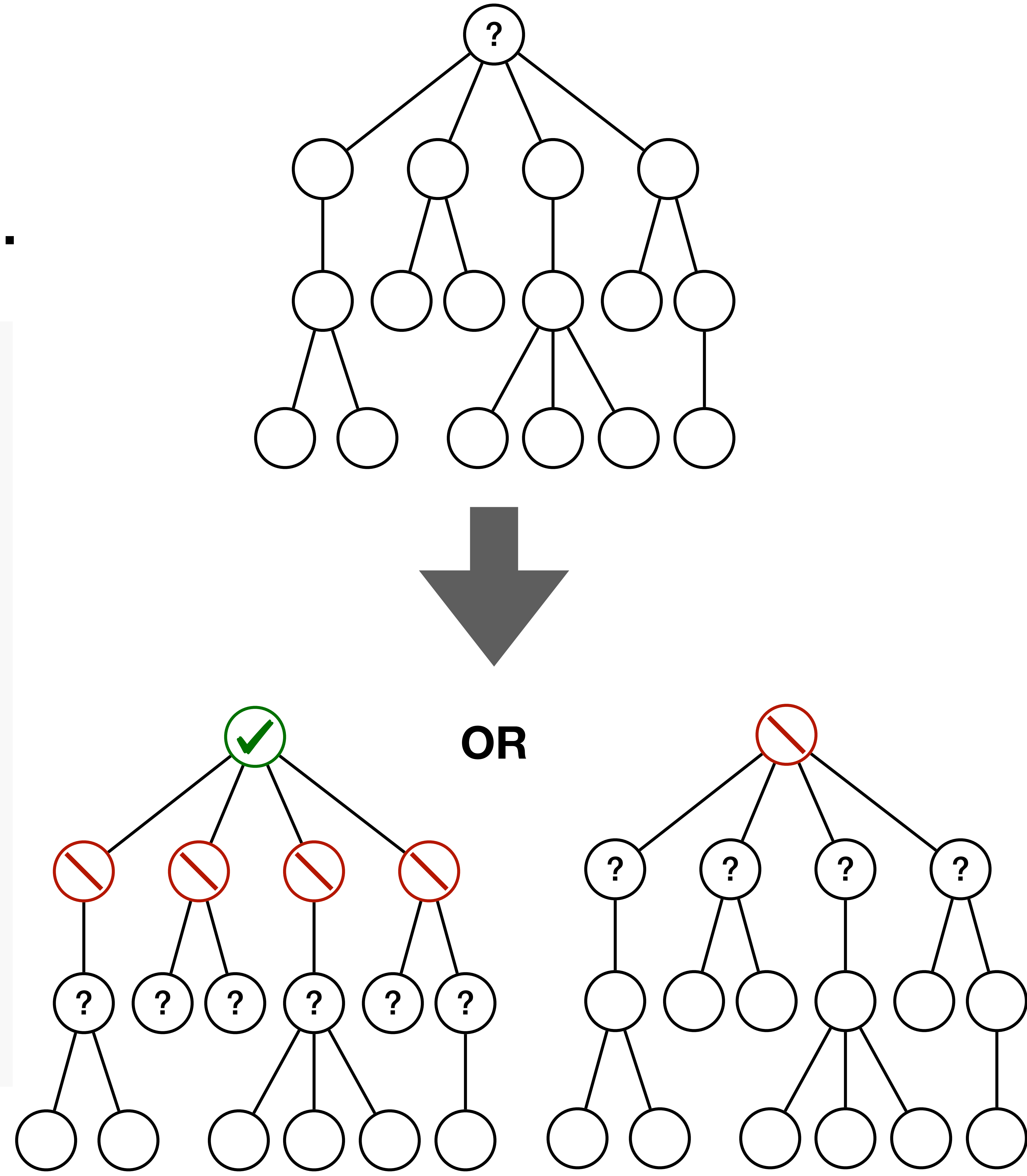
  $mis1 := mis1 + MaxISDP(v).mis0$

  $mis0 := mis0 + MaxISDP(v).mis$

$mis := Max(mis0, mis1)$

return $<mis, mis0, mis1>$

Runtime is $O(V+E) = O(V)$



Discussions of Dynamic Programming

Dynamic Programming (DP)

- Consider an (optimization) problem:
  - Build optimal solution step by step.
  - Problem has optimal substructure property.
    - We can design a recursive algorithm.
  - Problem has lots of overlapping subproblems.
    - Recursion and *memorize* solutions. (Top-Down)
    - Or, consider subproblems in the *right order*. (Bottom-Up)

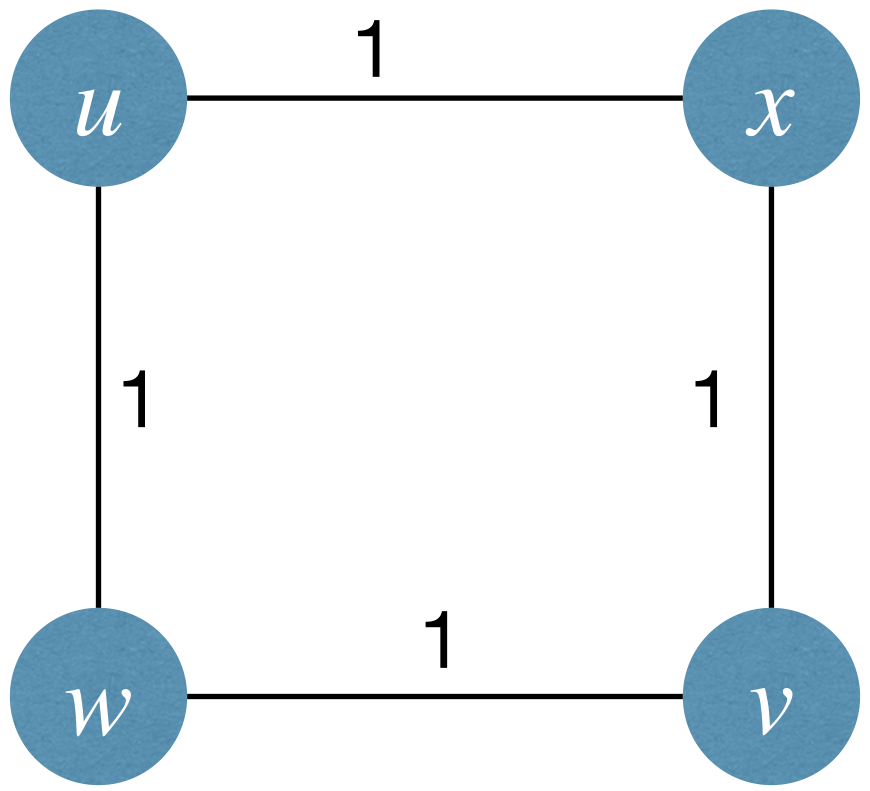
Optimal substructure not always true

Optimal substructure property!

Shortest path in unit-weight graph:

- Assume $w \in OPT(u \rightsquigarrow v)$
- $OPT(u \rightsquigarrow v) = u \overset{p_1}{\rightsquigarrow} w \overset{p_2}{\rightsquigarrow} v$
  - $p_1 = OPT(u \rightsquigarrow w)$
  - $p_2 = OPT(w \rightsquigarrow v)$

Subproblems are independent!



NO optimal substructure property!

- Longest *simple* path in unit-weight graph:
  - Assume $w \in OPT(u \rightsquigarrow v)$
  - $OPT(u \rightsquigarrow v) = u \overset{p_1}{\rightsquigarrow} w \overset{p_2}{\rightsquigarrow} v$
  - $p_1 = OPT(u \rightsquigarrow w)$?
    - Actually, $OPT(u \rightsquigarrow w) = u \rightsquigarrow x \rightsquigarrow v \rightsquigarrow w \neq p_1$
    - Similarly, $OPT(w \rightsquigarrow v) = w \rightsquigarrow u \rightsquigarrow x \rightsquigarrow v \neq p_2$

Subproblems are NOT independent!

Dynamic Programming (DP)

- Consider an (optimization) problem:
  - Build optimal solution step by step.
  - Problem has **optimal substructure** property.
    - We can design a recursive algorithm.
  - Problem has lots of **overlapping** subproblems.
    - Recursion and *memorize* solutions. (Top-Down)
    - Or, consider subproblems in the *right order*. (Bottom-Up)

Top-Down vs Bottom-Up

Dynamic programming trades space for time → Save solutions for subproblems to avoid repeat computation.

- [**Top-Down**] Recursion with memorization.
  - Very straightforward, easy to write down the code.
  - Use array or hash-table to memorize solutions.
  - Array may cost more space, but hash-table may cost more time.

- [**Bottom-Up**] Solve subproblems in the right order.
  - Finding the right order might be non-trivial. (Subproblem graph?)
  - Usually use array to memorize solutions.
  - Might be able to reduce the size of array to save even more space.

Top-down often costs more time in practice. (Recursion is costly!)
But not always! (Top-down only considers *necessary* subproblems.)

APSP via Dynamic Programming

$$dist(u,v,r) = \begin{cases} w(u,v) & if\ r = 0\ and\ (u,v) \in E \\ \infty & if\ r = 0\ and\ (u,v) \notin E \\ \min \left\{ \begin{array}{l} dist(u,v,r-1) \\ dist(u,x_r,r-1) + dist(x_r,v,r-1) \end{array} \right\} & otherwise \end{cases}$$

Space cost $O(n^3)$

FloydWarshallAPSP(G):

```
for each pair (u,v) in V*V
    if (u, v) in E then dist[u,v, 0] := w(u, v)
    else dist[u,v,0] := INF
for r := 1 to n
    for each node u
        for each node v
            dist[u,v,r] := dist[u,v,r - 1]
            if dist[u,v,r] > dist[u,x_r, r - 1] + dist[x_r,v, r - 1]
                dist[u,v,r] := dist[u,x_r, r - 1] + dist[x_r,v, r - 1]
```

Space cost $O(n^2)$

FloydWarshallAPSP(G):

```
for each pair (u,v) in V*V
    if (u, v) in E then dist[u,v] := w(u, v)
    else dist[u,v] := INF
for r := 1 to n
    for each node u
        for each node v
            if dist[u,v] > dist[u,x_r] + dist[x_r,v]
                dist[u,v] := dist[u,x_r] + dist[x_r,v]
```

Edit Distance

$$dist(i,j) = \begin{cases} i & if\ j = 0 \\ j & if\ i = 0 \\ \min \begin{Bmatrix} dist(i, j-1) + 1 \\ dist(i-1, j) + 1 \\ dist(i-1, j-1) + I[A[i] = B[j]] \end{Bmatrix} & otherwise \end{cases}$$

EditDistDP(A[1…m],B[1…n]):

Space cost $O(n^2)$

for $i := 0$ **to** m
  $dist[i, 0] := i$
for $j := 0$ **to** n
  $dist[0, j] := j$
for $i := 1$ **to** m
  for $j := 1$ **to** n
    $delDist := dist[i-1, j] + 1$
    $insDist := dist[i, j-1] + 1$
    $subDist := dist[i-1, j-1] + Diff(A[i], B[j])$
    $dist[i, j] := Min(delDist, insDist, subDist)$
return $dist$

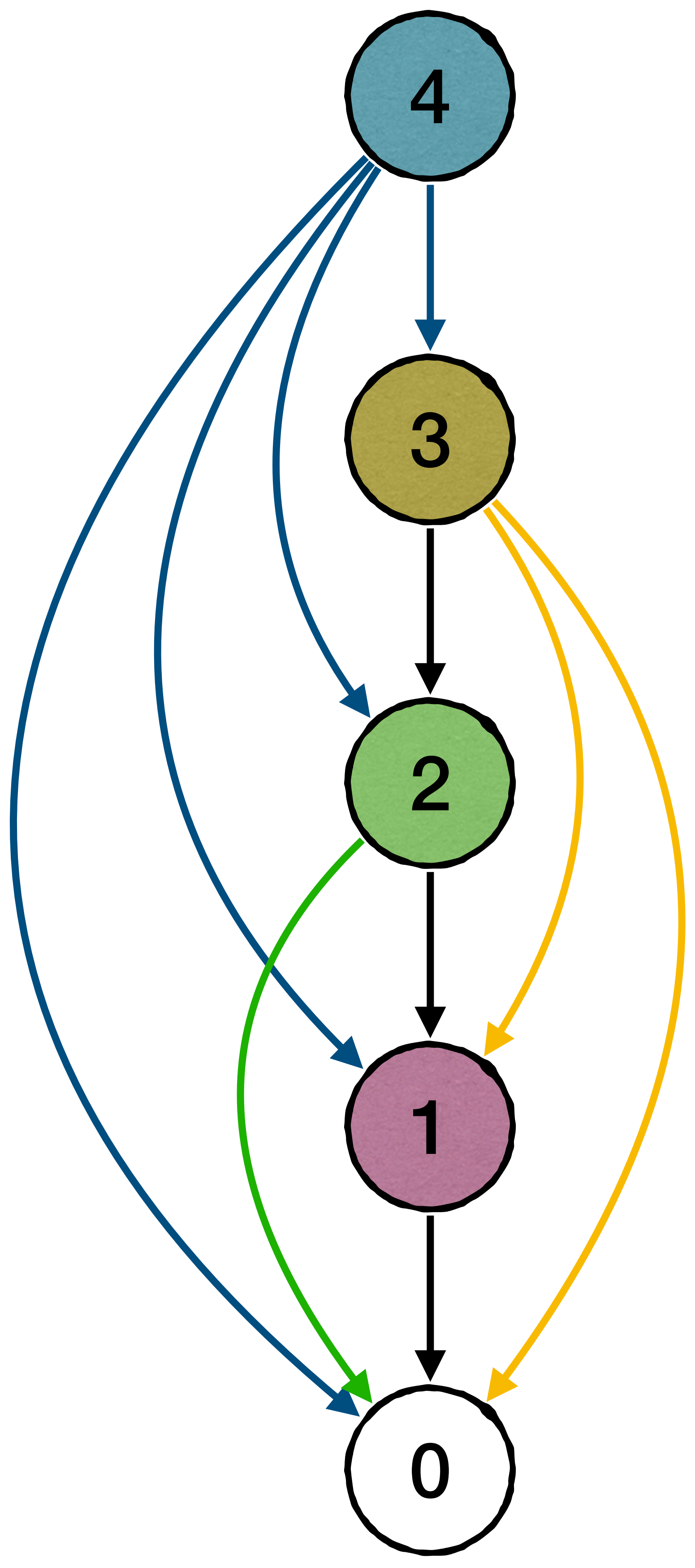
EditDistDP(A[1…m],B[1…n]):

Space cost $O(n)$

for $j := 0$ **to** n
  $distLast[j] := j$ $//distLast[j] = dist[i-1, j]$
for $i := 0$ **to** m
  $distCur[0] := i$ $//distCur[j] = dist[i, j]$
  for $j := 1$ **to** n
    $delDist := distLast[j] + 1$
    $insDist := distCur[j-1] + 1$
    $subDist := distLast[j-1] + Diff(A[i], B[j])$
    $distCur[j] := Min(delDist, insDist, subDist)$
  $distLast := distCur$
return $distCur[n]$

Analysis of DP Algorithms

- Correctness:
  - Optimal substructure property.
  - **Bottom-up approach:** subproblems are already solved.
- Complexity:
  - **Space complexity:** usually obvious.
  - Time complexity [bottom-up]: usually obvious.
  - Time complexity [top-down]:
    - How many subproblems in total?(number of nodes in the subproblem DAG.)
    - Time to solve a problem, given subproblem solutions?(number of edges in the subproblem DAG.)



Subset Sum

- **Problem:** Given an array $X[1 \cdots n]$ of n positive integers, can we find a subset in X that sums to given integer T?

- **Simple solution:** recursively enumerates all 2^n subsets, leading to an algorithm costing $O(2^n)$ time.

- Can we do better with dynamic programming?(Notice this is **not** an optimization problem.)

Subset Sum

- **Problem:** Given an array $X[1 \cdots n]$ of n **positive** integers, can we find a subset in X that sums to given integer T?

- **Step 1**: Characterize the structure of solution.

  - If there is a solution S, either $X[1]$ is in it or not.

  - If $X[1] \in S$, then there is a solution to instance "$X[2...n], T - X[1]$";

  - If $X[1] \notin S$, then there is a solution to instance "$X[2...n], T$".

Subset Sum

- Step 2: Recursively define the value of an optimal solution.
  - Let $ss(i, t)$ = true iff instance "$X[i \ldots n], t$" has a solution.
  - $$ss(i,t) = \begin{cases} true & \text{if } t = 0 \\ ss(i+1, t) & \text{if } t < X[i] \\ false & \text{if } i > n \\ ss(i+1, t) \vee ss(i+1, t - X[i]) & \text{otherwise} \end{cases}$$

Subset Sum

- $$ss(i,t) = \begin{cases} true & \text{if } t = 0 \\ ss(i+1,t) & \text{if } t < X[i] \\ false & \text{if } i > n \\ ss(i+1,t) \vee ss(i+1,t-X[i]) & \text{otherwise} \end{cases}$$

- Step 3: Compute the value of an optimal solution (Bottom-Up).
  - Build an 2D array $ss[1...n, 0...T]$
  - Evaluation order: bottom row to top row; left to right within each row.

Runtime is $O(nT)$

<u>SubsetSumDP(X,T):</u>

```
ss[n,0] := True
for t := 1 to T
    ss[n,t] := (X[n] = t) ? True : False
for i := n - 1 downto 1
    ss[i,0] := True
    for t :=1 to X[i] - 1
        ss[i,t] := ss[i + 1, t]
    for t := X[i] to T
        ss[i,t] := Or(ss[i + 1, t], ss[i + 1, t - X[i]])
return ss[1,T]
```

Subset Sum

- **Problem:** Given an array $X[1 \cdots n]$ of n positive integers, can we find a subset in X that sums to given integer T?
- **Simple solution:** recursively enumerates all 2^n subsets, leading to an algorithm costing $O(2^n)$ time.
- Dynamic programming: costing $O(nT)$ time.
  - Dynamic programming isn't *always* an improvement! (Depends on T)

Dynamic Programming vs Greedy

Common strategies for solving optimization problems → Gradually generates a solution for the problem

- **Dynamic Programming**
  - **At each step**: *multiple potential* choices, each reducing the problem to a subproblem, compute *optimal solutions of all subproblems* and then find optimal solution of original problem.
  - Optimal substructure + ***Overlapping subproblems.***

- **Greedy**
  - **At each step**: make *an optimal* choice, then compute *optimal solution of the subproblem* induced by the choice made.
  - Optimal substructure + ***Greedy choice***

Try DP first, then check if greedy works! (If does, prove it!)
(Come up with a working algorithm first, then develop a faster one.)

Further reading

- [CLRS] Ch.1
- [DPV] Ch.6
- [Erickson] Ch.3