分治策略
Divide and Conquer

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The slides are mainly adapted from the original ones shared by Chaodong Zheng and Kevin Wayne. Thanks for their supports! We also use some materials from stanford-cs161.

The Divide-and-Conquer Approach

- **Divide** the given problem into a number of subproblems that are smaller instances of the same problem.
- **Conquer** the subproblems by solving them recursively.
  - Or, use brute-force if a subproblem is small enough.
- **Combine** the solutions for the subproblems to obtain the solution for the original problem.

Described in pseudocode

Solve (I):

if I *is small enough:*

$solution := DirectSolve(I)$

else

$< I_1, I_2, \ldots, I_k > := DivideProblem(I)$

for $j := 1$ *to* k

$solution_j = Solve(I_j)$

$solution = Combine(solution_1, \ldots, solution_k)$

return solution

Direct solve the basic case, or use brute-force if (sub)problem is simple

Divide the problem into **smaller** subproblems.

Recursively solve subproblems.

Combine solutions of subproblems to get solution for original problem.

Correctness of Divide-and-Conquer

- How to prove the correctness of a divide-and-conquer algorithm?
  - Use (strong) mathematical induction, proceeding by induction on the "size" of the inputs.
- **Induction basis**: prove the algorithm can correctly solve small problem instances.
  - Prove `DirectSolve` is correct if $|I| \leq c$.
- **Induction hypothesis**: the algorithm can correctly solve **any** problem instance of size at most, say, n.
  - `Solve` is correct if $|I| \leq n$.
- **Inductive step**: assuming induction hypothesis, prove the algorithm can correctly solve problem instance of size $n+1$.
  - Assume `Solve` is correct if $|I| \leq n$, Prove `Solve` is correct if $|I| = n+1$

Solve (I):

if I *is small enough:*
 solution := *DirectSolve(I)*
else
 $< I_1, I_2, \ldots, I_k >$:= *DivideProblem(I)*
 for $j := 1$ *to* k
  $solution_j = Solve(I_j)$
 $solution = Combine(solution_1, \ldots, solution_k)$
return solution

Partial or **Total** Correctness?

Termination and partial correctness can be encapsulated !

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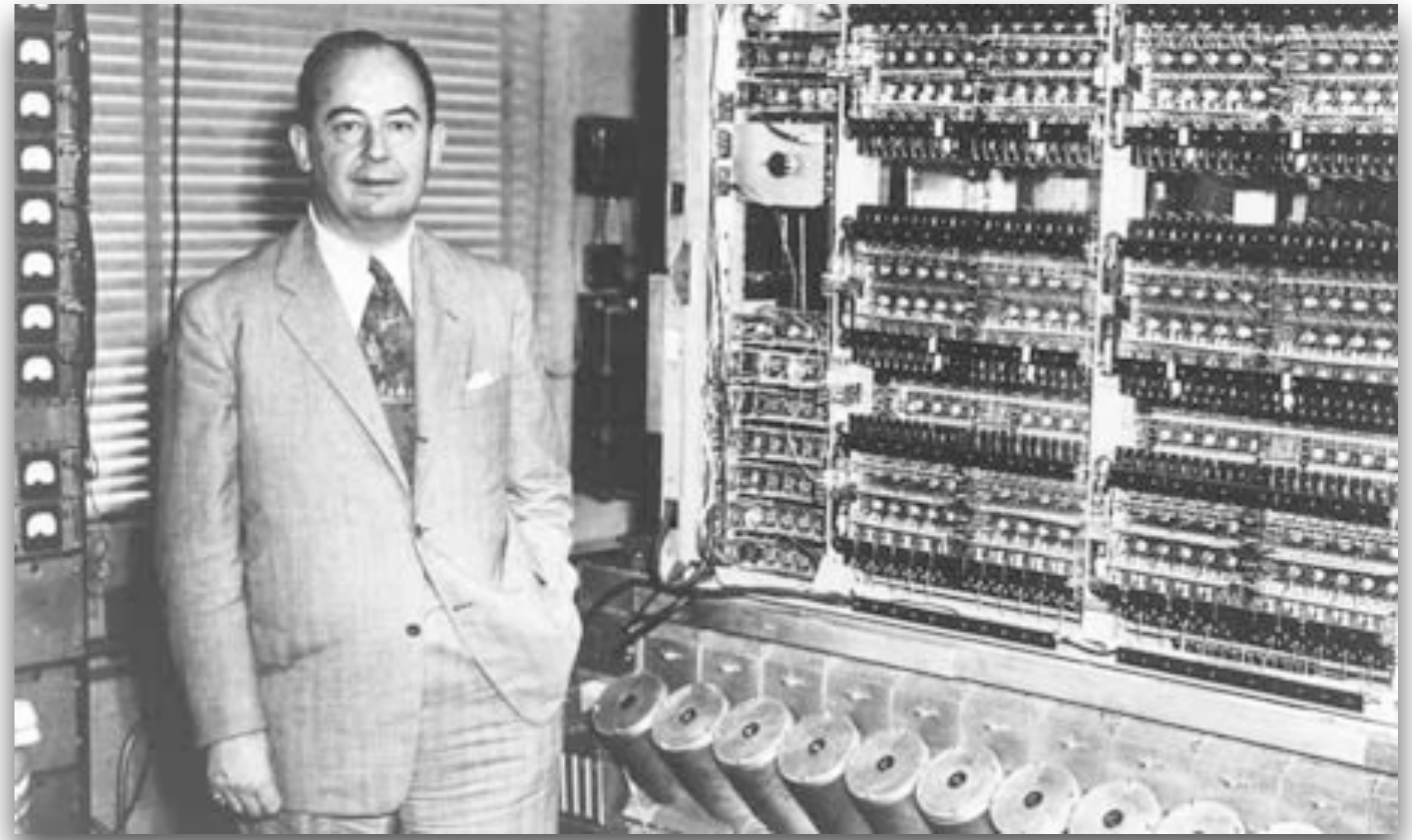
Merge Sort

MergeSort

- An efficient divide-and-conquer algorithm for sorting.
- Invented by John von Neumann in the 1940s.



MergeSort

Divide-and-Conquer Template

Solve (I):

if I is small enough:

  $solution := DirectSolve(I)$

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  $< I_1, I_2, \ldots, I_k > := DivideProblem(I)$

  for $j := 1$ to k

    $solution_j = Solve(I_j)$

  $solution = Combine(solution_1, \ldots, solution_k)$

return solution

MergeSort ($A[1\ldots n]$):

if $n = 1$:

  $sol[1\ldots n] := [1\ldots n]$

else

  $solLeft[1\ldots(n/2)] := MergeSort(A[1\ldots(n/2)])$

  $solRright[1\ldots(n/2)] := MergeSort(A[(n/2+1)\ldots n])$

  $sol[1\ldots n] := Merge(solLeft[1\ldots(n/2)], solRight[1\ldots(n/2)])$

return $sol[1\ldots n]$

Merge ($A[1\ldots n], B[1\ldots m]$):

$Aindex := 1, Bindex := 1, Result := []$

// Scan A and B from left to right,

// Append the currently smallest to the result array

while $Aindex \leq A.length$ **and** $Bindex \leq B.length$

  if $A[Aindex] \leq B[Aindex]$

    $Result.AddLast(A[Aindex])$

    $Aindex := Aindex + 1$

  else

    $Result.AddLast(B[Bindex])$

    $Bindex := Bindex + 1$

// Copy the remaining elements of A and B

while $Aindex \leq A.length$

  $Result.AddLast(A[Aindex])$

  $Aindex := Aindex + 1$

while $Bindex \leq B.length$

  $Result.AddLast(B[Bindex])$

  $Bindex := Bindex + 1$

return $Result$

The Merge Subroutine

```
Merge ( A[1…n], B[1…m]):
  Aindex := 1, Bindex := 1,  Result := []

 // Scan A and B from left to right,
 //  Append the currently smallest to the result array
  while Aindex ≤ A.length  and  Bindex ≤ B.length
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// Copy the remaining elements of  A and B
     while  Aindex ≤ A.length
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      return Result
```

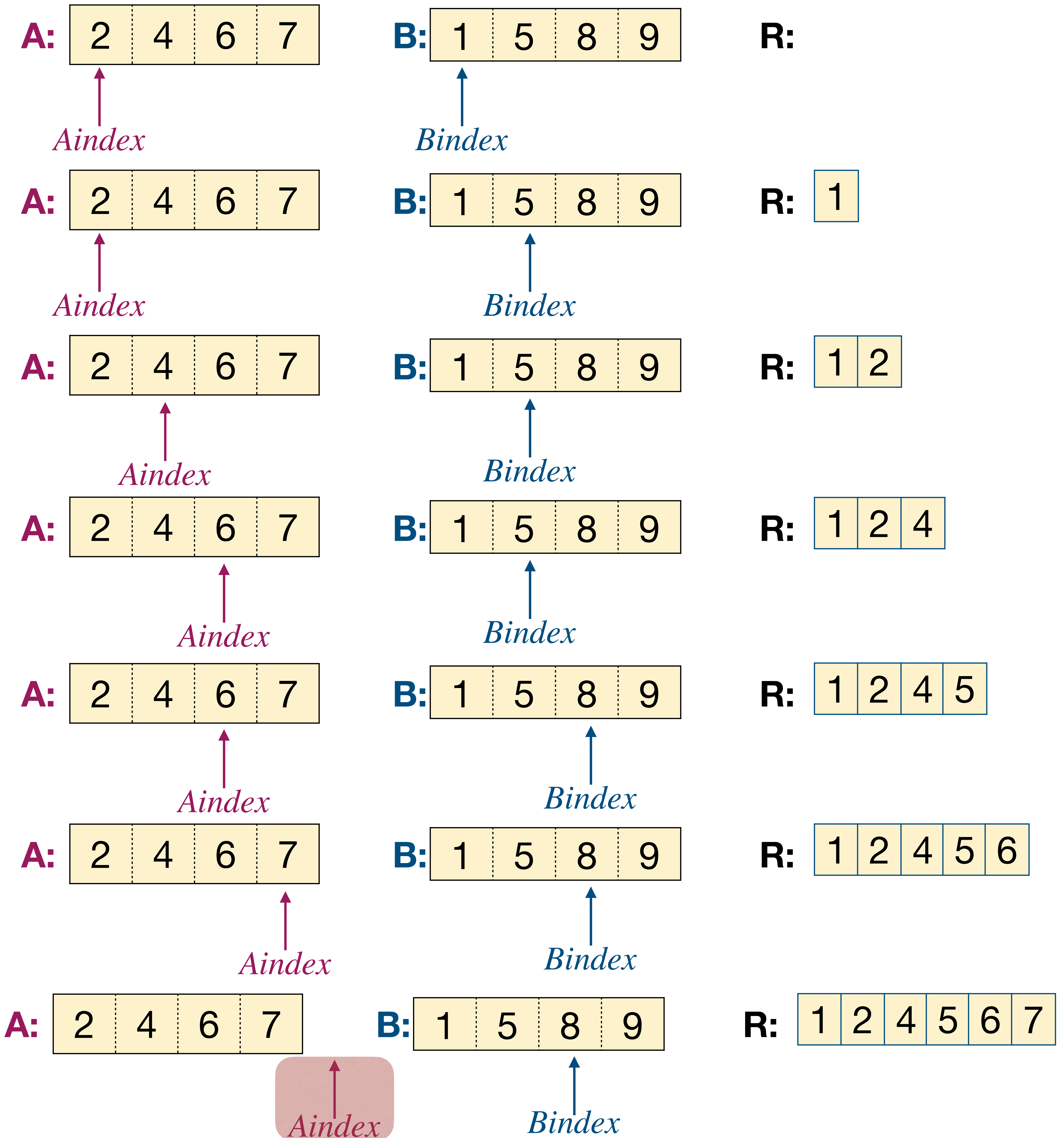

The Merge Subroutine

Merge ($A[1\ldots n]$, $B[1\ldots m]$):

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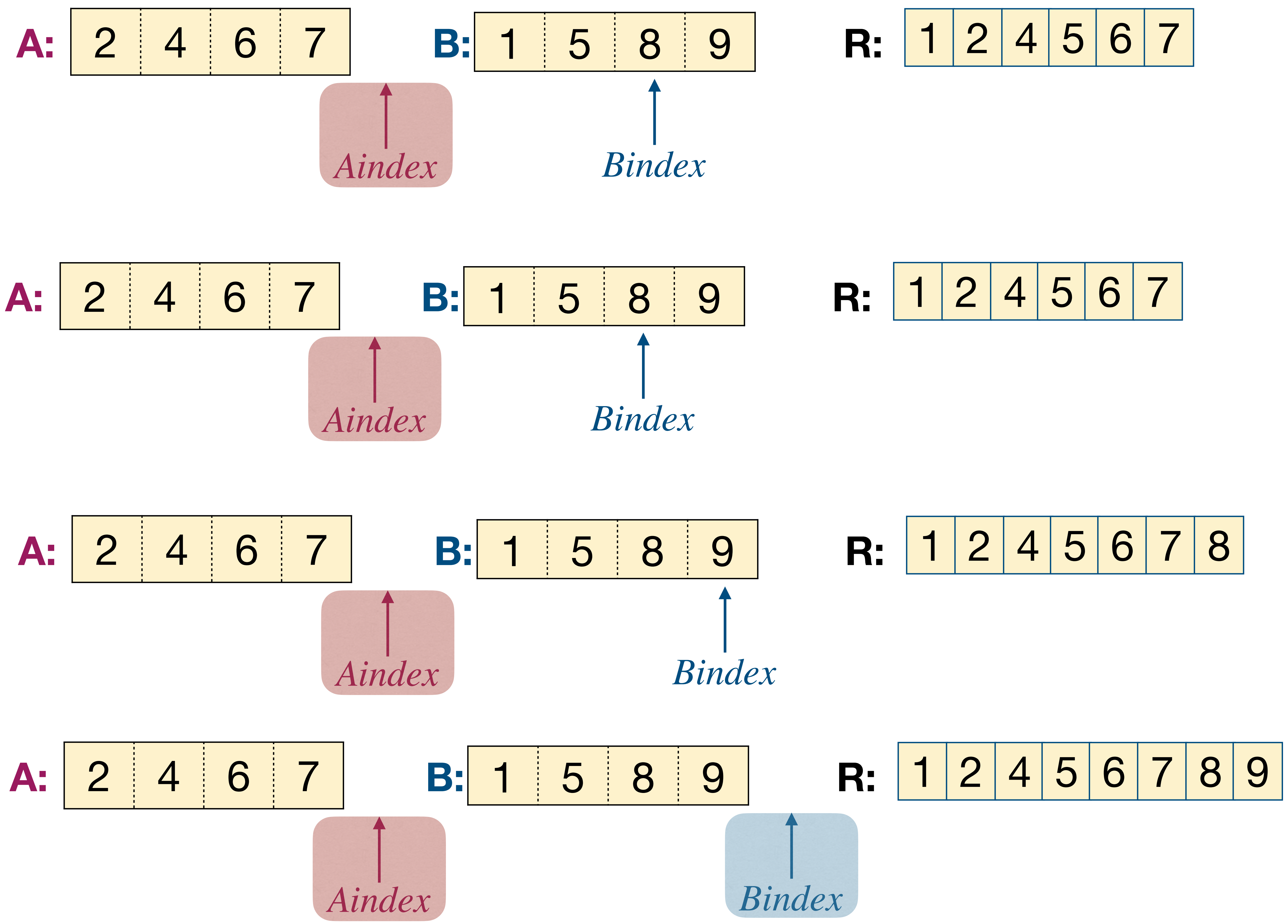

Sample execution of MergeSort

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  $sol[1\ldots n] := Merge(solLeft[1\ldots(n/2)], solRight[1\ldots(n/2)])$
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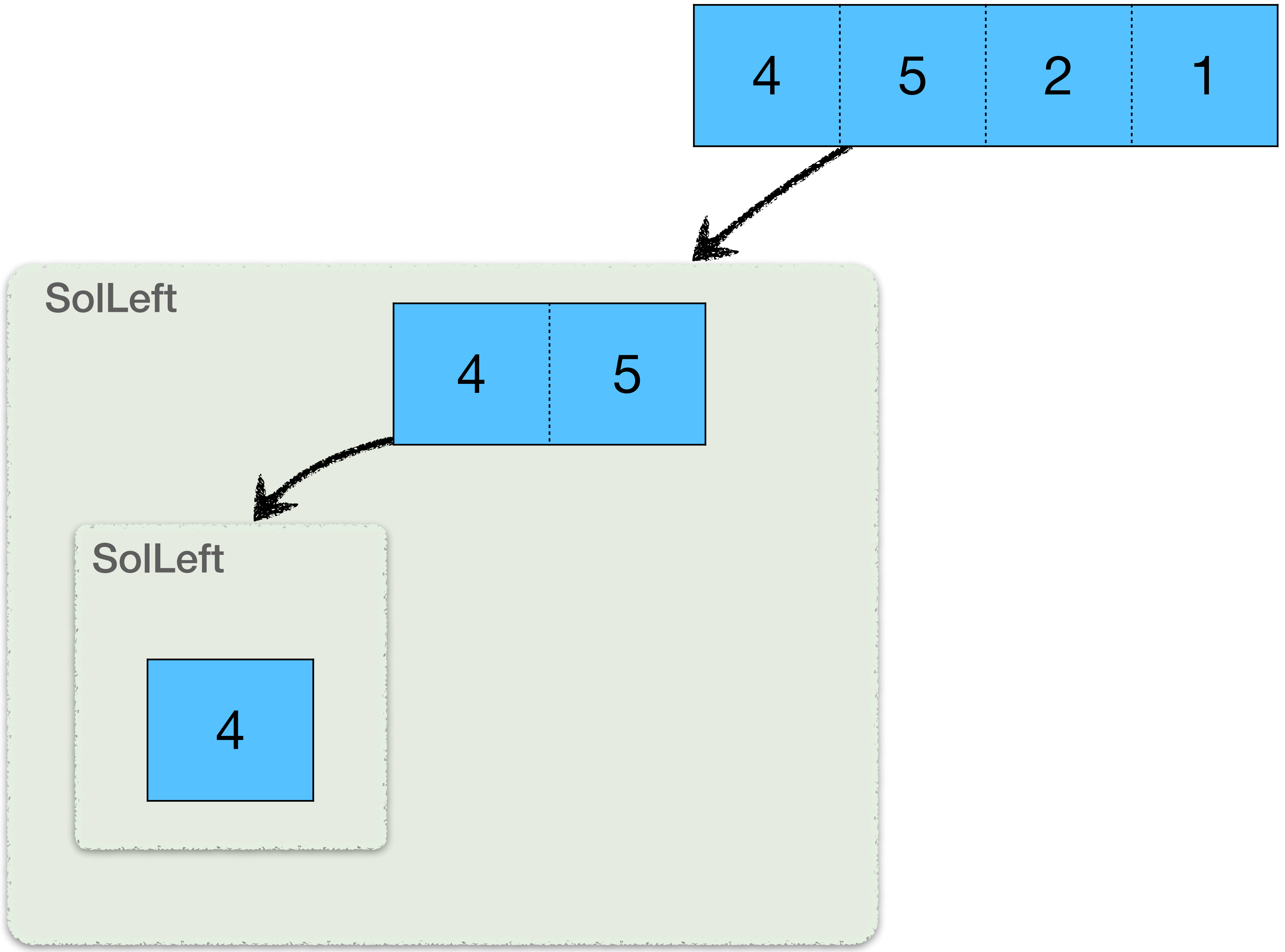

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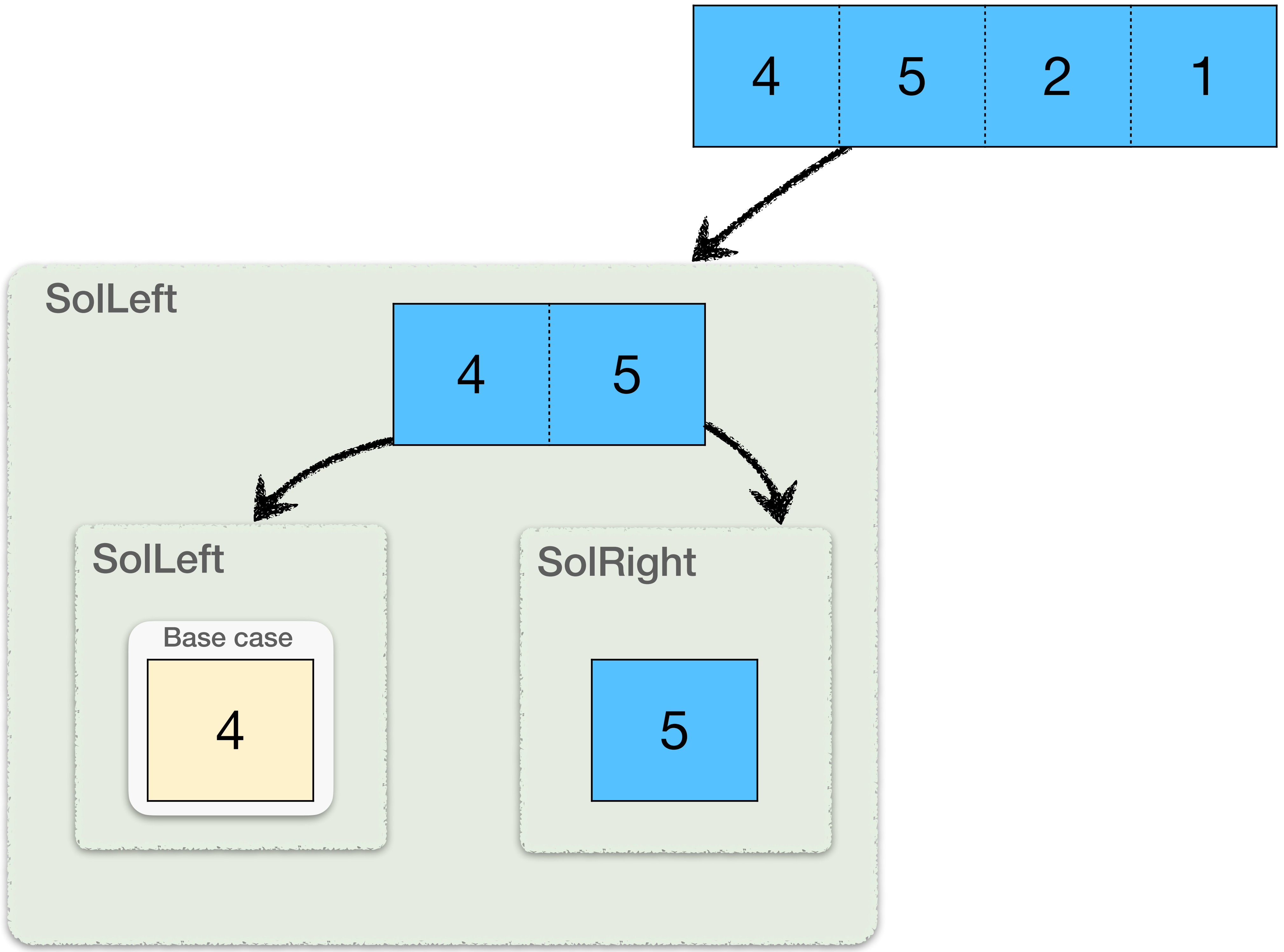

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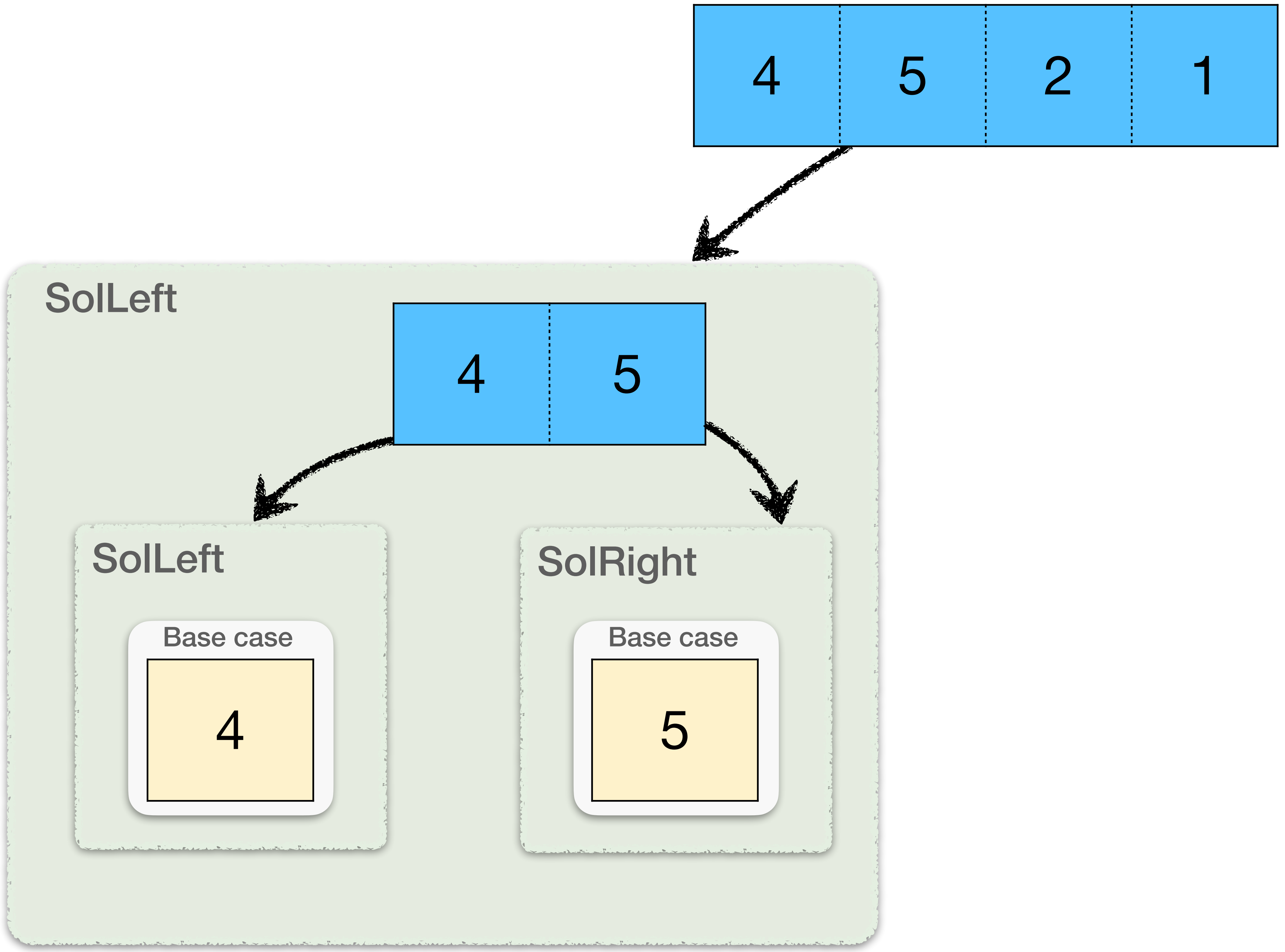

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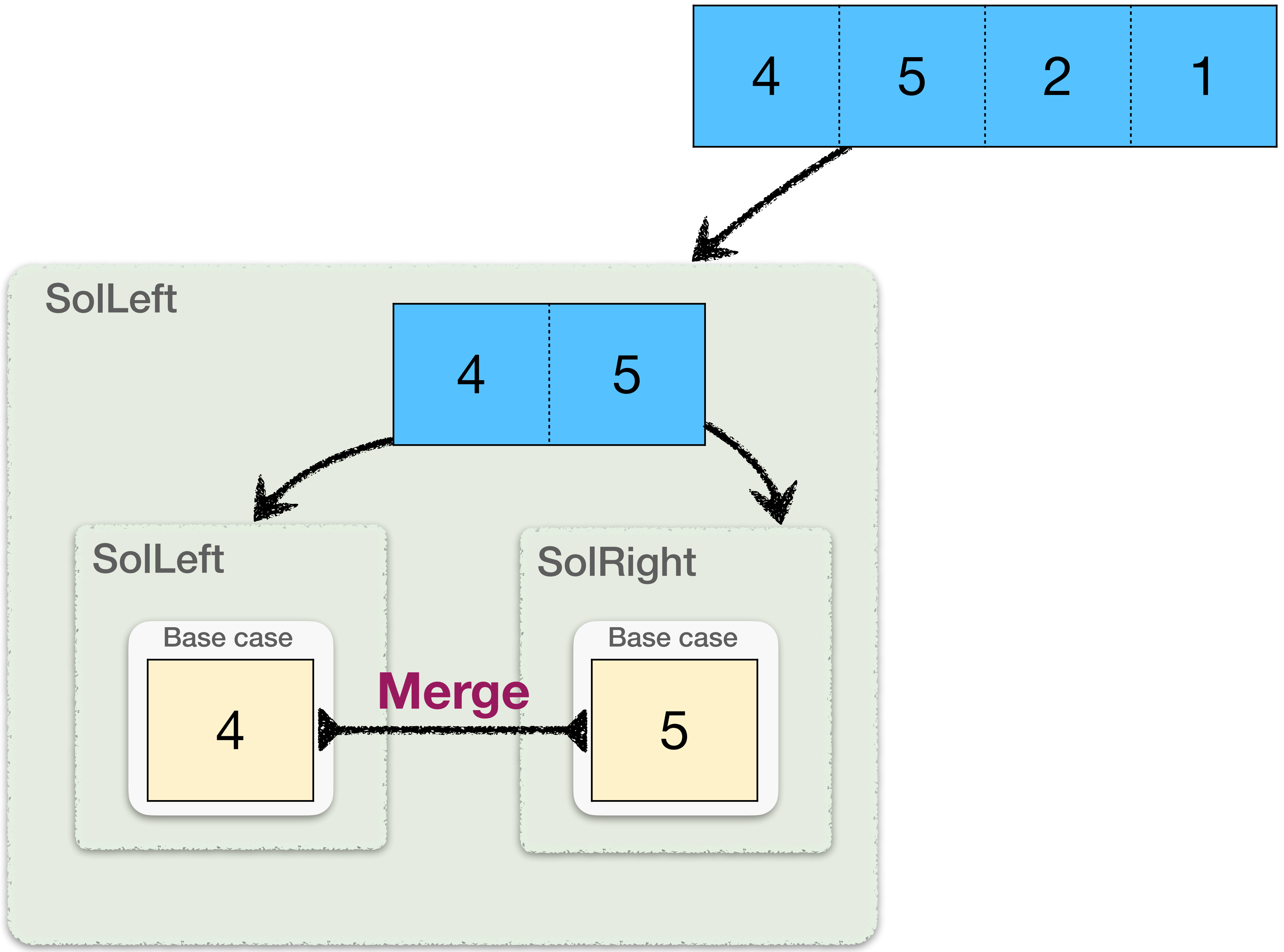

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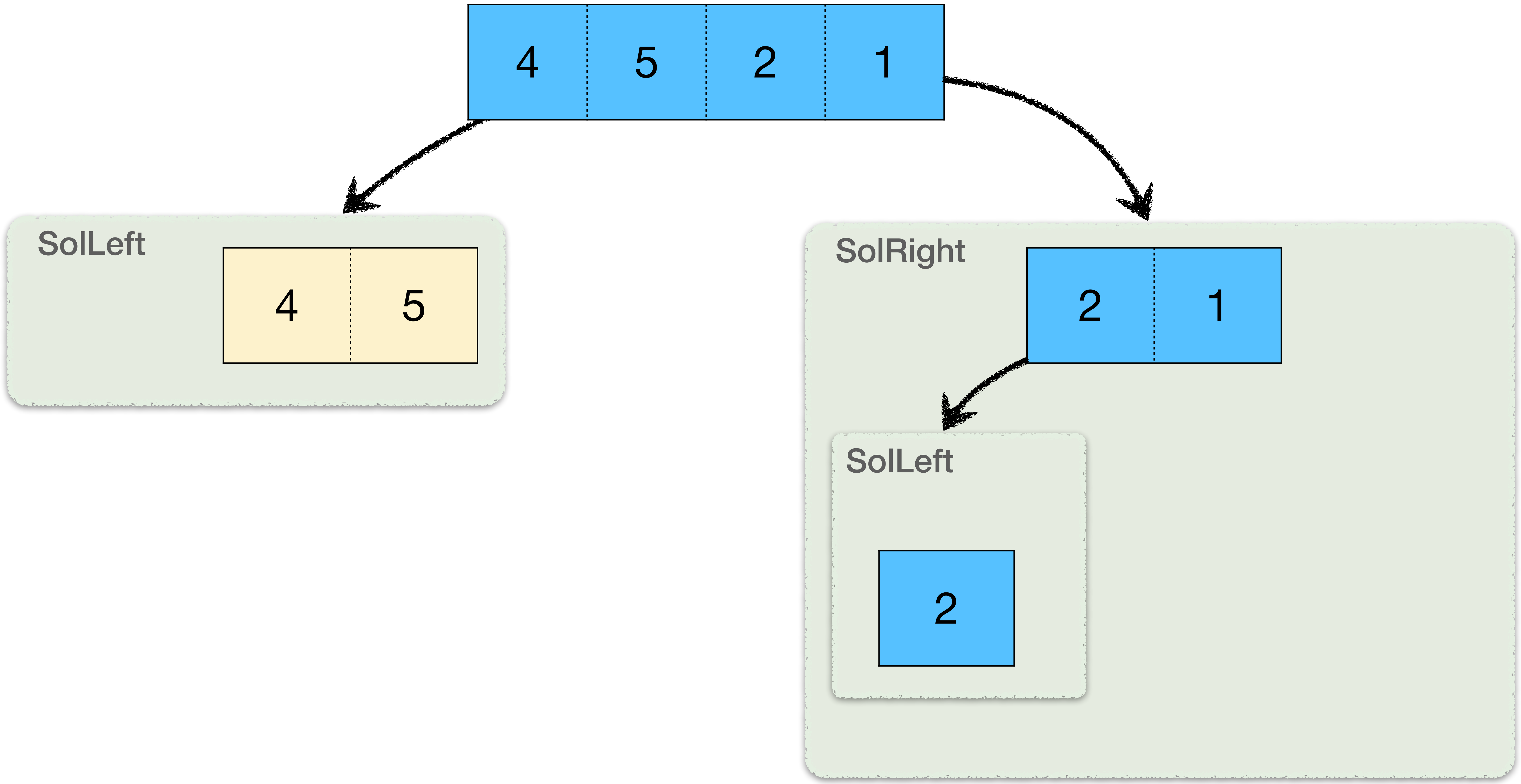

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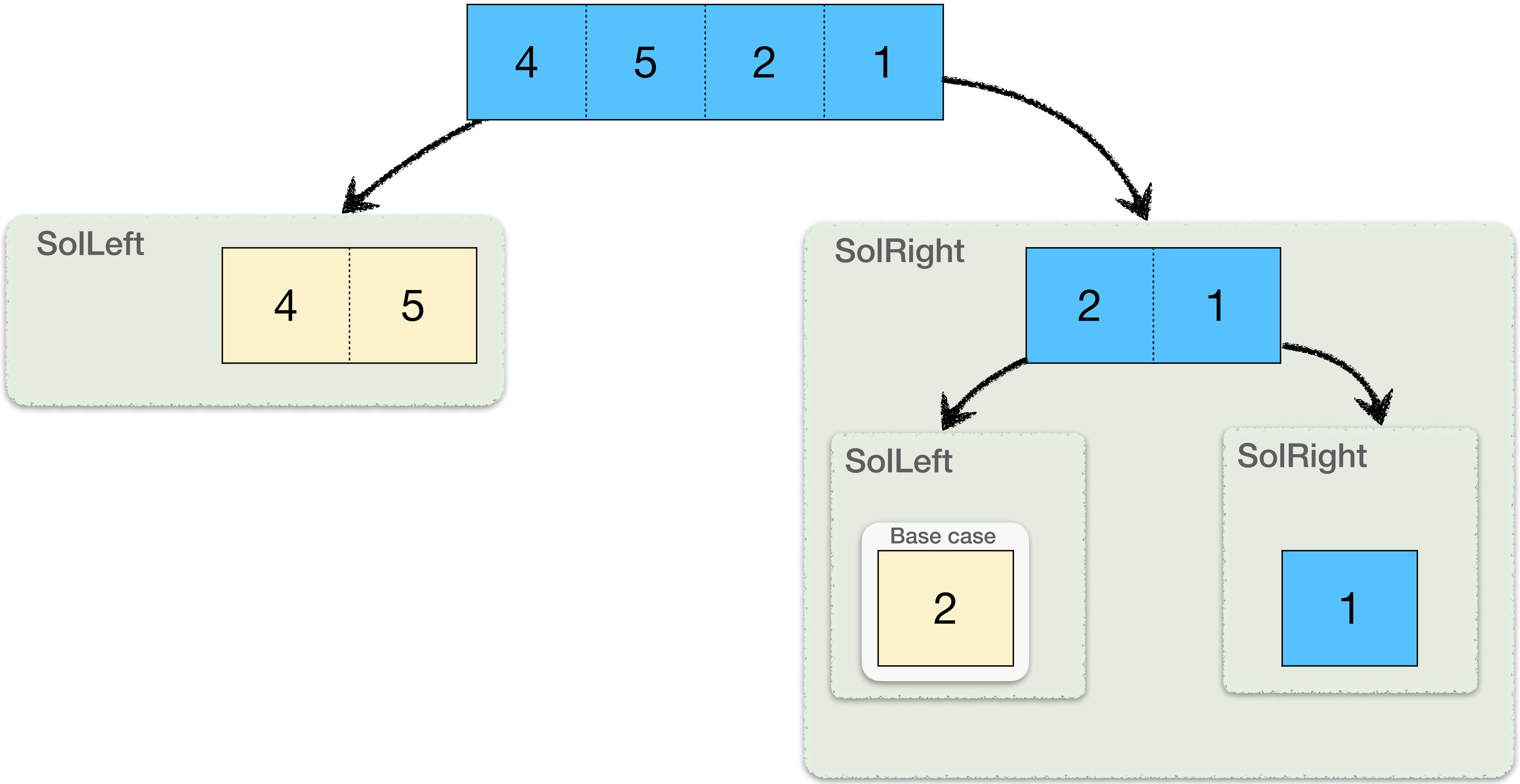

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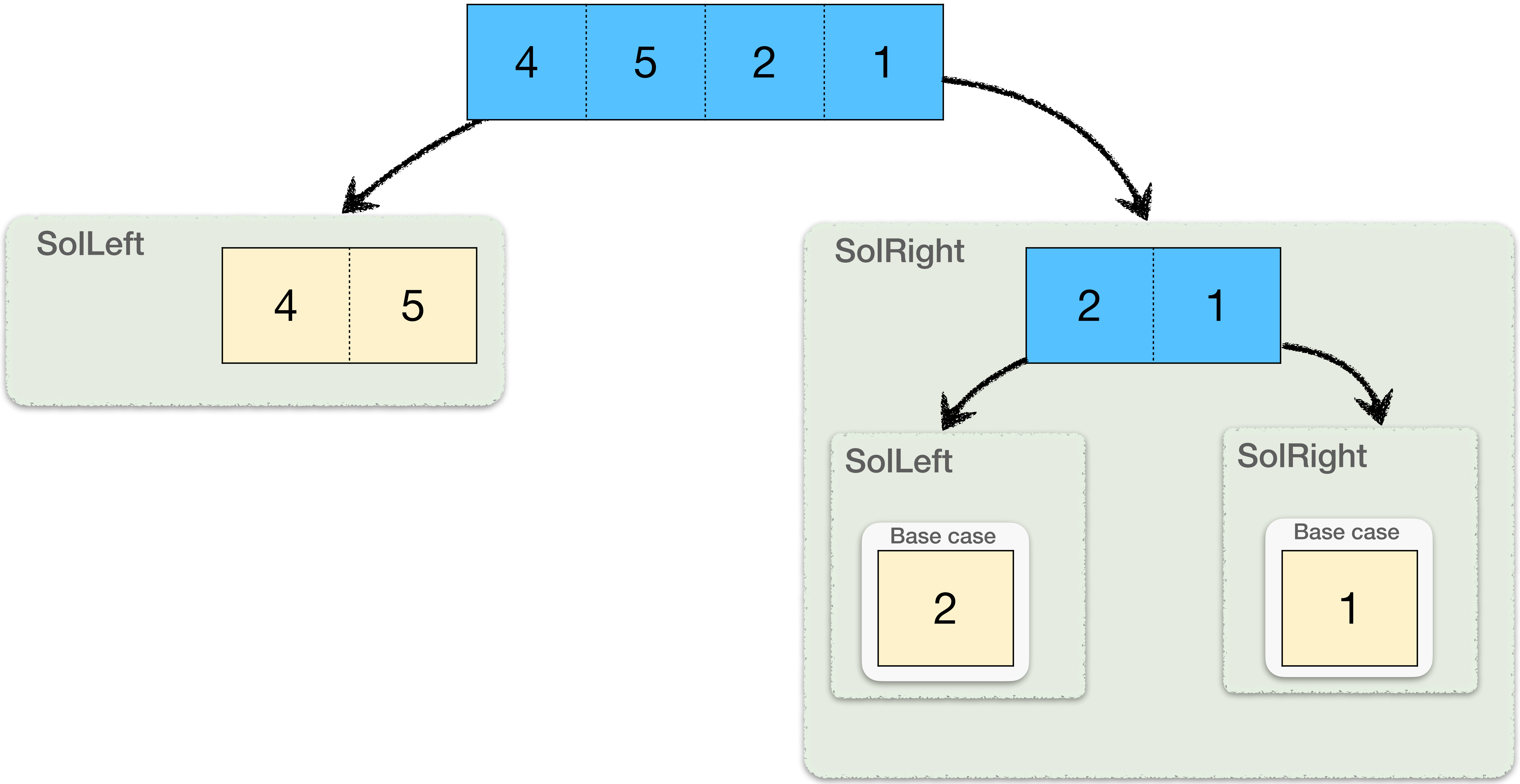

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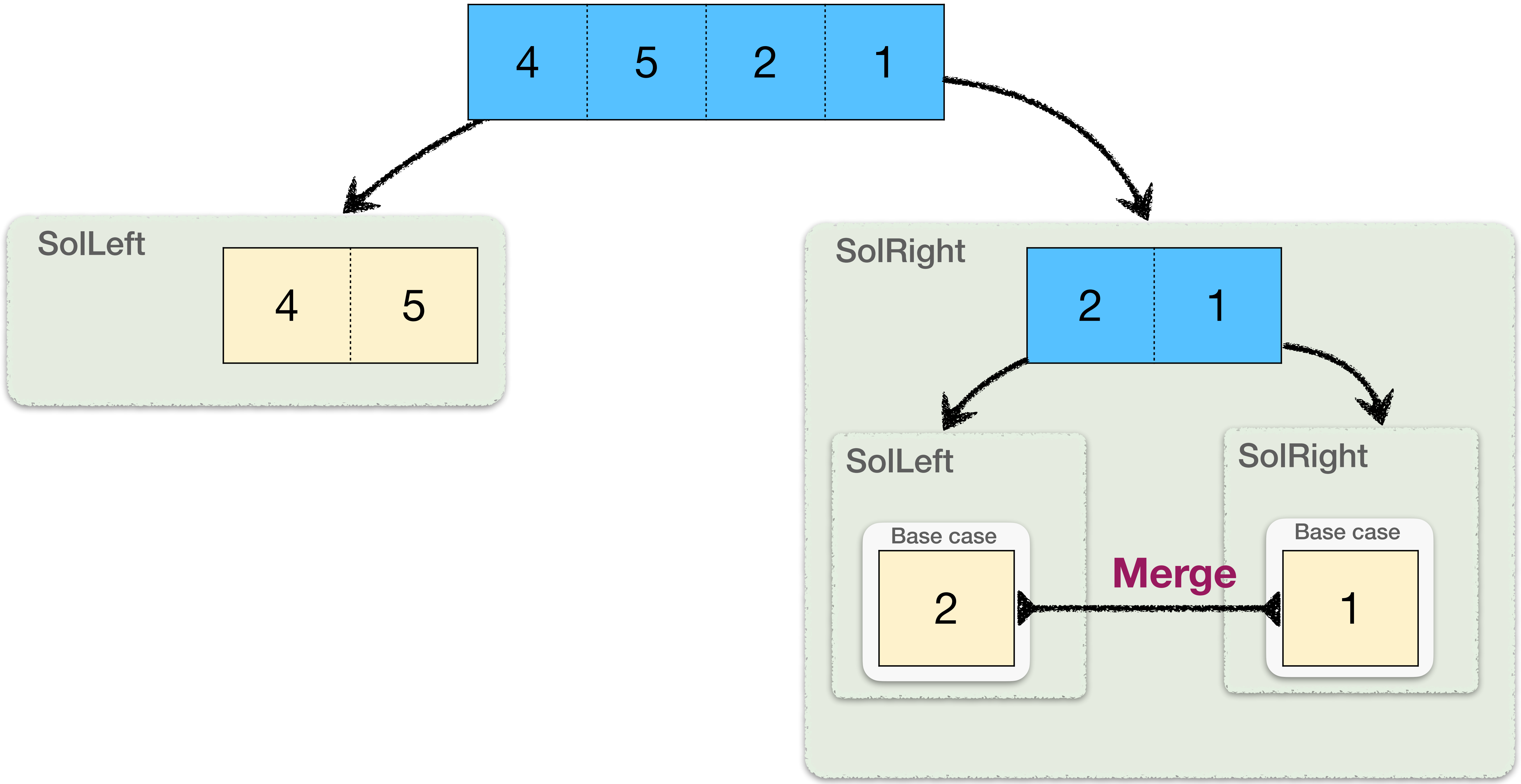
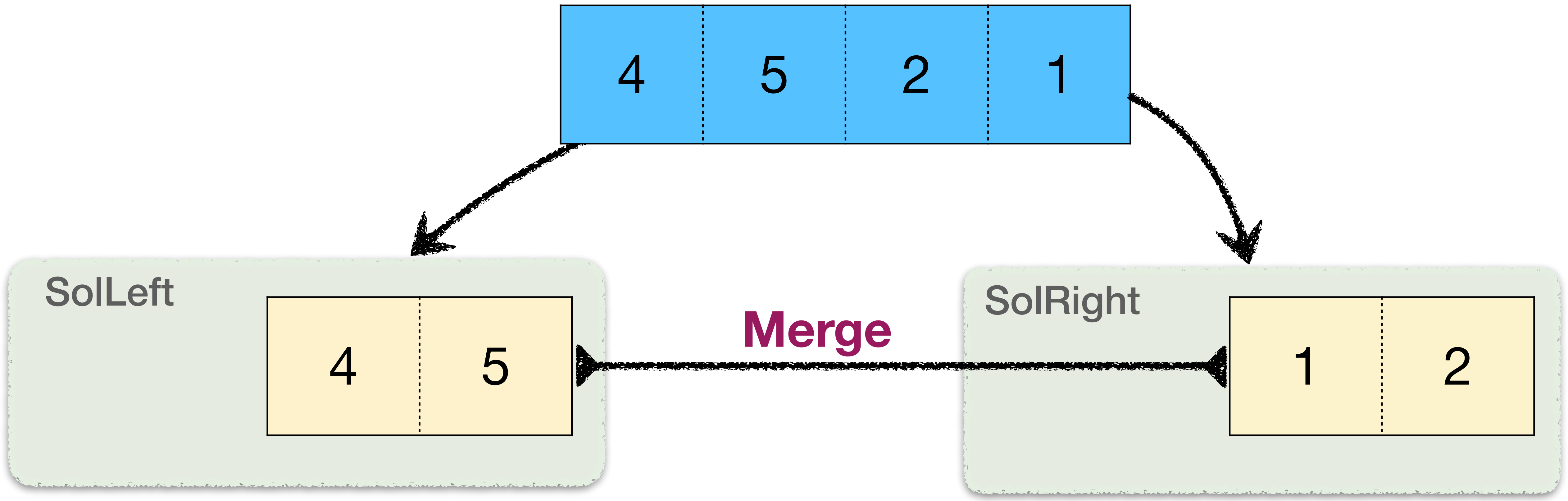


Sample execution of MergeSort

MergeSort ($A[1...n]$):
if $n = 1$:
  $sol[1...n]$:= $[1...n]$
else
  $solLeft[1...(n/2)] := MergeSort(A[1...(n/2)])$
  $solRright[1...(n/2)] := MergeSort(A[(n/2+1)...n])$
  $sol[1...n] := Merge(solLeft[1...(n/2)], solRight[1...(n/2)])$
return $sol[1...n]$



Sample execution of MergeSort

MergeSort ($A[1\ldots n]$):

```
if n = 1:
    sol[1…n] := [1…n]
else
    solLeft[1…(n/2)] := MergeSort(A[1…(n/2)])
    solRright[1…(n/2)] := MergeSort(A[(n/2+1)…n])
    sol[1…n] := Merge(solLeft[1…(n/2)], solRight[1…(n/2)])
return sol[1…n]
```

1	2	4	5

Correctness of MergeSort

```
MergeSort ( A[1…n]):
  if n = 1:
      sol[1…n] := [1…n]
  else
      solLeft[1…(n/2)] := MergeSort(A[1…(n/2)])
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      sol[1…n] := Merge(solLeft[1…(n/2)], solRight[1…(n/2)])
  return sol[1…n]
```

- **Induction basis**: `MergeSort` is correct when $n = 1$.
- **Induction hypothesis**: Assume `MergeSort` is correct if $n \leq n'$
- **Inductive step**: `MergeSort` is correct when $n = n' + 1$

How to prove the correctness of the subroutine?

- Correctness of this routine?
  - Find proper loop invariant!
  - What is it?

Merge ($A[1\ldots n], B[1\ldots m]$):

```
  Aindex := 1, Bindex := 1,  Result := []

 // Scan A and B from left to right,
 //  Append the currently smallest to the result array
  while Aindex ≤ A.length  and  Bindex ≤ B.length
     if A[Aindex] ≤ B[Aindex]
        Result.AddLast(A[Aindex])
        Aindex := Aindex + 1
     else
        Result.AddLast(B[Bindex])
        Bindex := Bindex + 1

// Copy the remaining elements of  A and B
    while  Aindex ≤ A.length
        Result.AddLast(A[Aindex])
        Aindex := Aindex + 1
    while  Bindex ≤ B.length
        Result.AddLast(B[Bindex])
        Bindex := Bindex + 1
     return Result
```

Time complexity of MergeSort

MergeSort ($A[1\ldots n]$):

```
if n = 1:
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Merge ($A[1\ldots n]$, $B[1\ldots m]$):

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// Copy the remaining elements of A and B
while Aindex ≤ A.length
    Result.AddLast(A[Aindex])
    Aindex := Aindex + 1
while Bindex ≤ B.length
    Result.AddLast(B[Bindex])
    Bindex := Bindex + 1
return Result
```

- For Subroutine *Merge*, the four "*while*" processes involves scanning all the elements in A and B.
- The " *if* " processes has fewer comparisons than "*while*" processes
- Therefore, the time complexity of Subroutine *Merge* is $\Theta(n)$, where n is the sum of the elements of A and B.

Time complexity of MergeSort

```
MergeSort ( A[1…n]):
  if n = 1:
     sol[1…n] := [1…n]
  else
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     solRright[1…(n/2)] := MergeSort(A[(n/2+1)…n])
      sol[1…n] := Merge(solLeft[1…(n/2)], solRight[1…(n/2)])
  return sol[1…n]
```

- For the main procedure `MergeSort`:
  - Let $T(n)$ be the runtime of `MergeSort` on instance of size n.
  - Clearly, $T(1) = c_1 = \Theta(1)$ for some constant c_1.
  - For larger n, $T(n) = 2 \cdot T(n/2) + c_2 \cdot n = 2T(n/2) + \Theta(n)$.

Time complexity of **MergeSort**

- A **recurrence** equation:
  - $$\begin{cases} T(1) = c_1 \\ T(n) = 2 \cdot T(n/2) + c_2 \cdot n \end{cases}$$

$T(n)$

Time complexity of MergeSort

- A **recurrence** equation:
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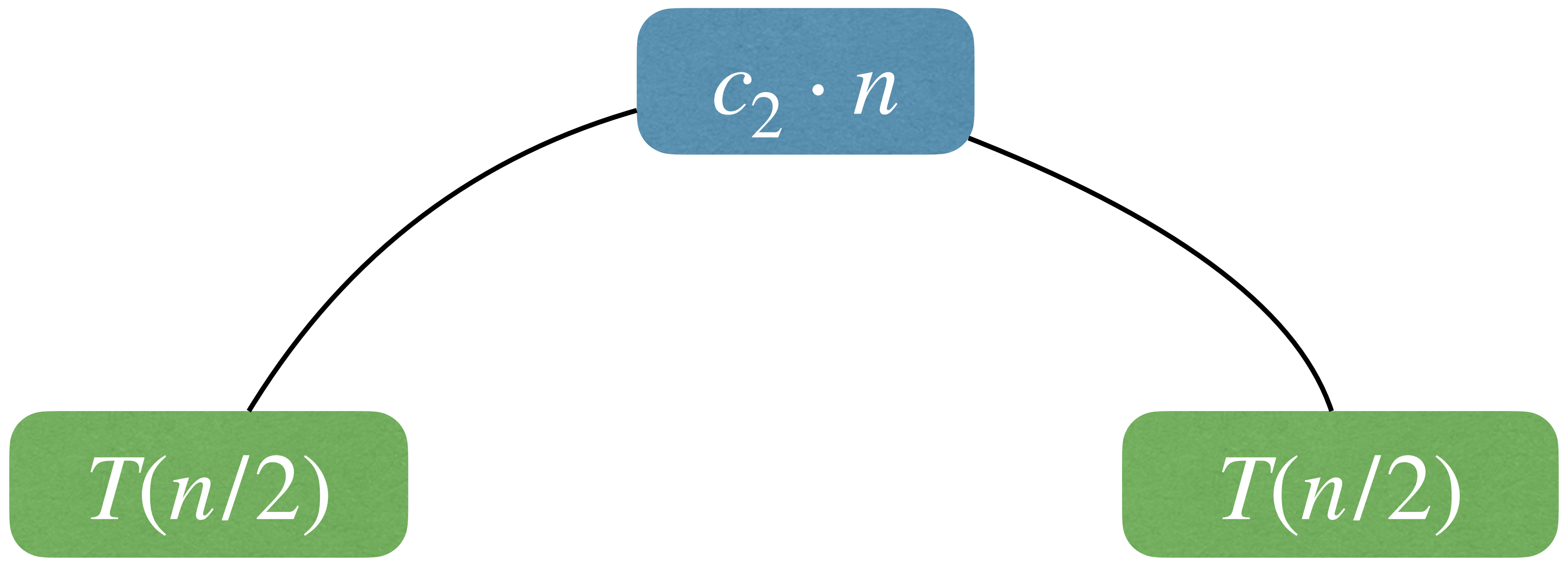

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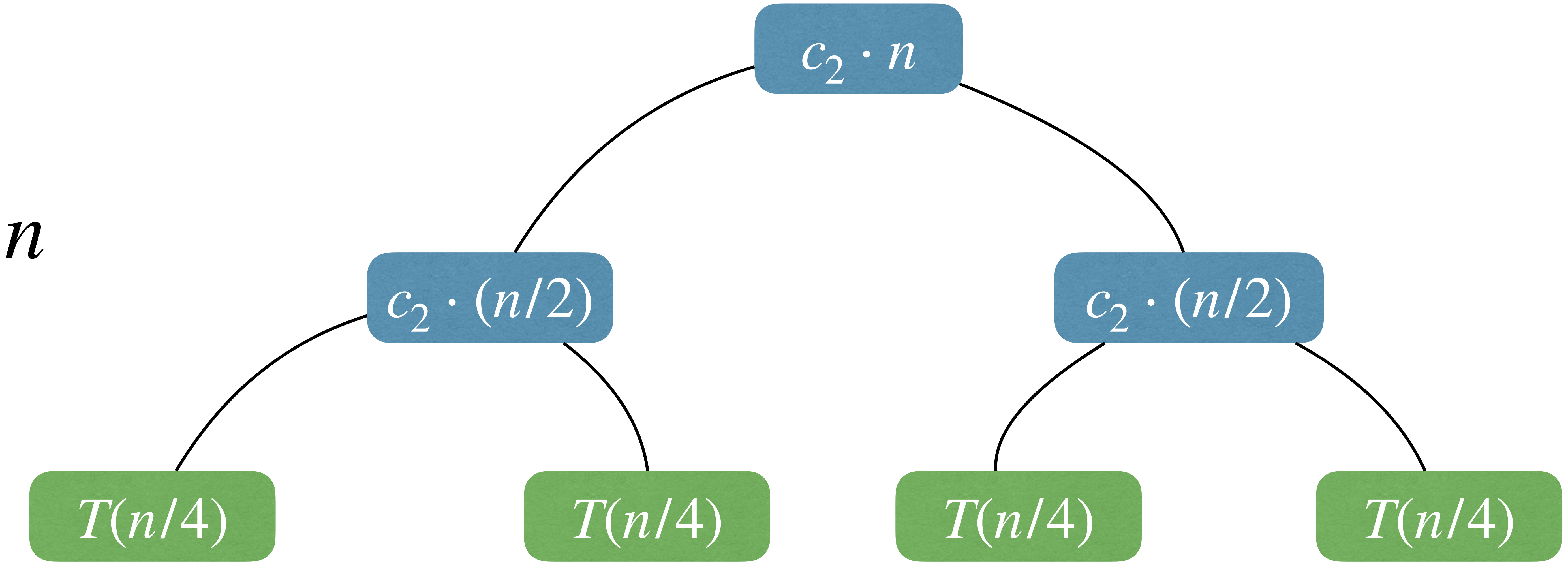

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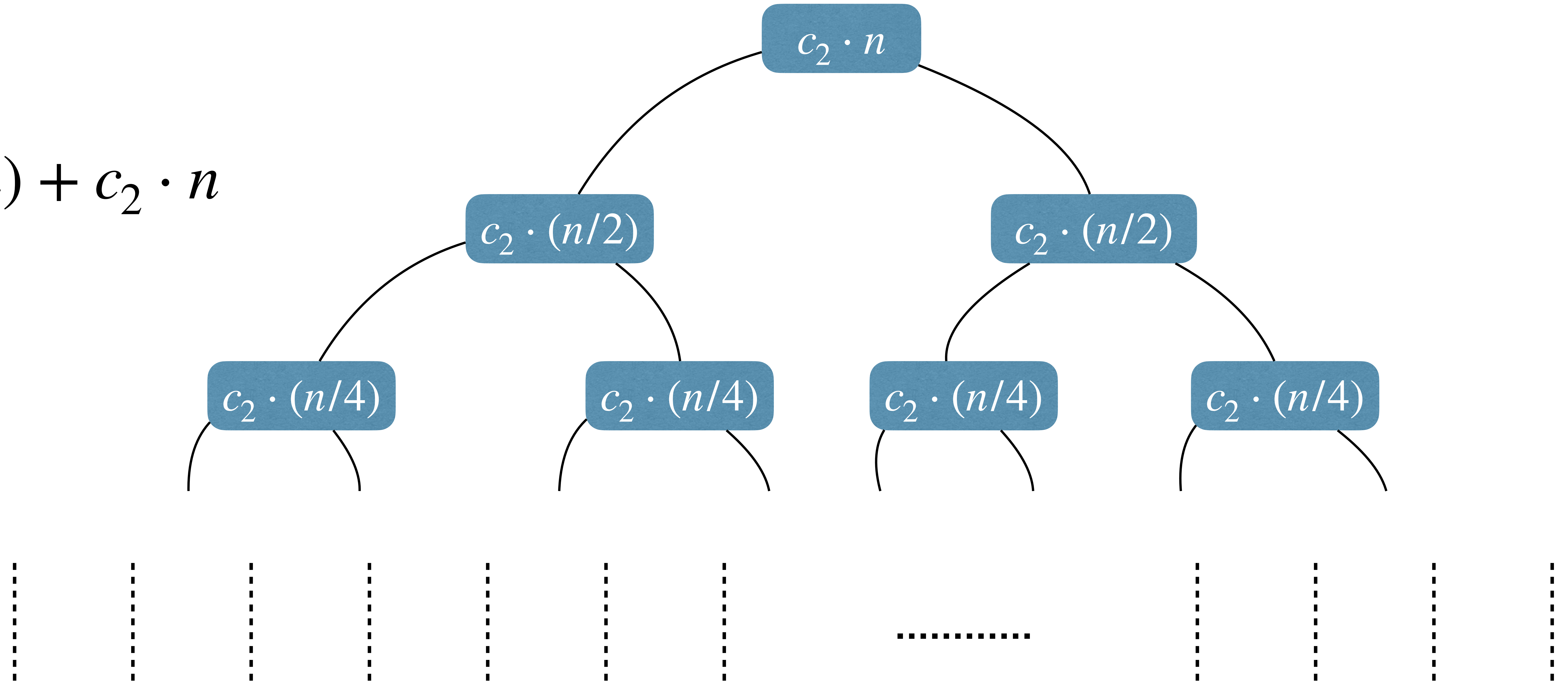
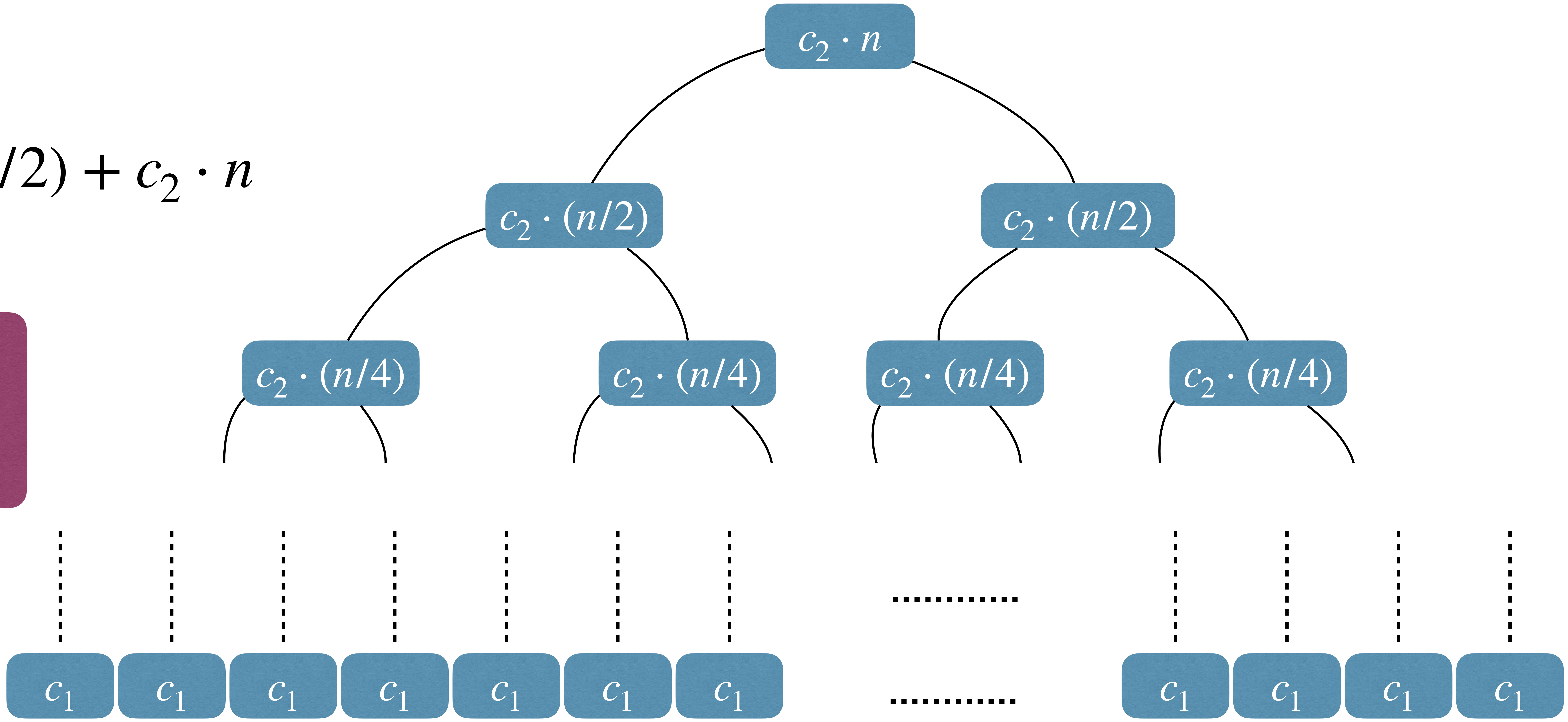


Time complexity of MergeSort

- A **recurrence** equation:
  - $$\begin{cases} T(1) = c_1 \\ T(n) = 2 \cdot T(n/2) + c_2 \cdot n \end{cases}$$

There are $\log_2 n$ + 1 levels
Each level incur $\Theta(n)$
Total cost is $\Theta(n \cdot \log_2 n)$



Recursion tree

Iterative MergeSort

MergeSort ($A[1 \ldots n]$):

if $n = 1$:

$sol[1 \ldots n]$:= $[1 \ldots n]$

else

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$solRright[1 \ldots (n/2)] := MergeSort(A[(n/2+1) \ldots n])$

$sol[1 \ldots n] := Merge(solLeft[1 \ldots (n/2)], solRight[1 \ldots (n/2)])$

return $sol[1 \ldots n]$

- Any recursive algorithm can be converted into an iterative one, we just simulate the call stack!

Iterative MergeSort

IterMergeSort ($A[1\ldots n]$):

$Deque\ Q_1, Q_2$

for $i := 1$ **to** n

  $Q_1.addLast(A[i])$

while true

  while $Q_1.size() > 1$

    $L := Q_1.removeFirst(),\ R := Q_1.removeFirst()$

    $Q_2.AddLast(Merge(L, R))$

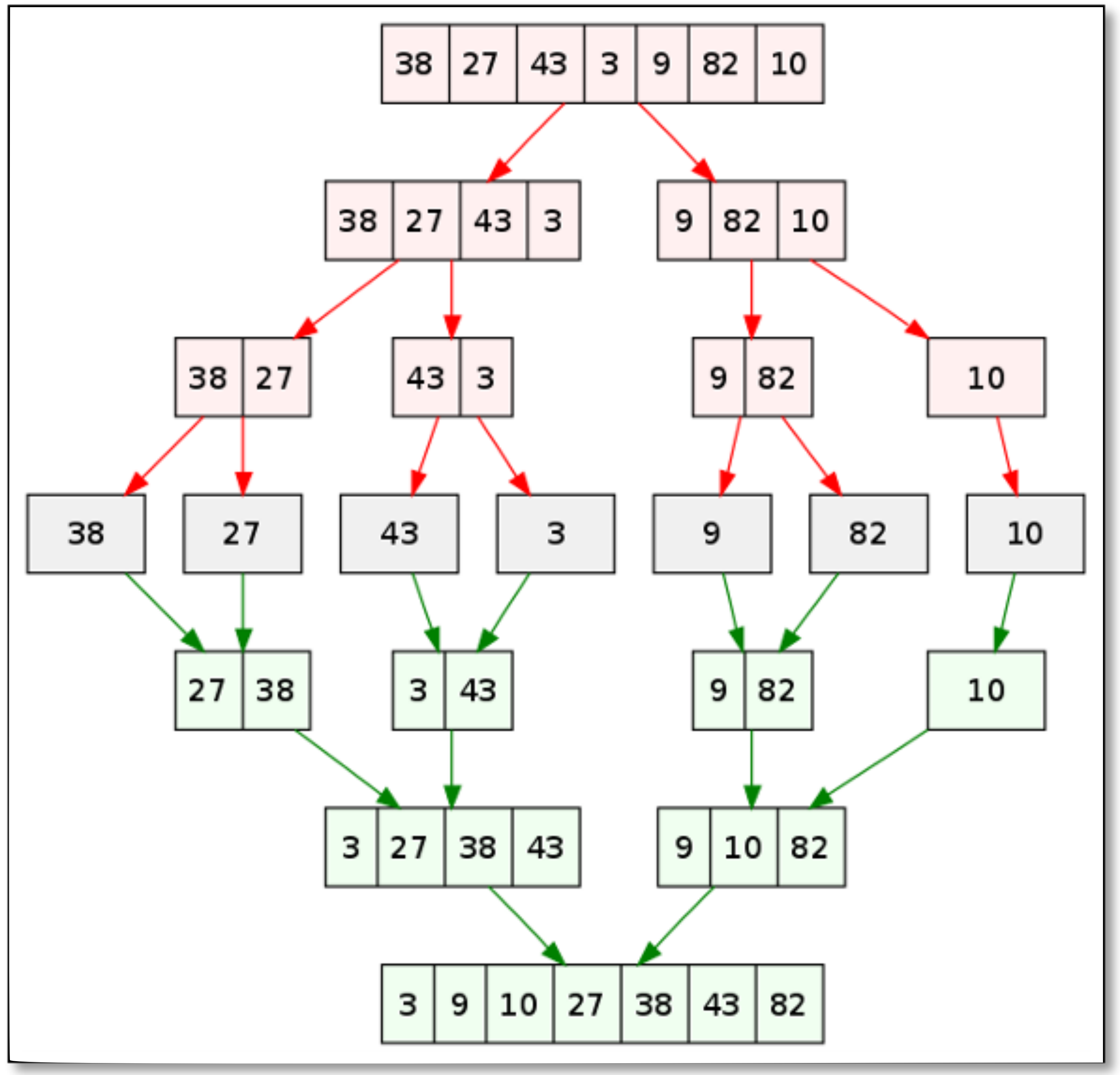
  $Q_2.AddLast(Q_1.removeFirst())$

  $Q_1 := Q_2$

  if $Q_1.size() = 1$

    break

return $Q.removeFirst()$



Do "Merge" operation layer by layer!

The time complexity is $\Theta(n \cdot \log n)$

Matrix Multiplication

Matrix Multiplication

- Suppose we want to multiply two $n \times n$ matrices $\mathbf{X}$ and $\mathbf{Y}$.
- The most straightforward method needs $\Theta(n^3)$ time.
- Matrix multiplication can be performed block-wise!
  - $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$
  - $XY = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$

Matrix Multiplication

- $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$
- $XY = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{bmatrix}$
- The recurrence equation is $T(n) = 8 \cdot T(n/2) + \Theta(n^2)$
- Thus, $T(n) = \Theta(n^3)$, which has no improvement…

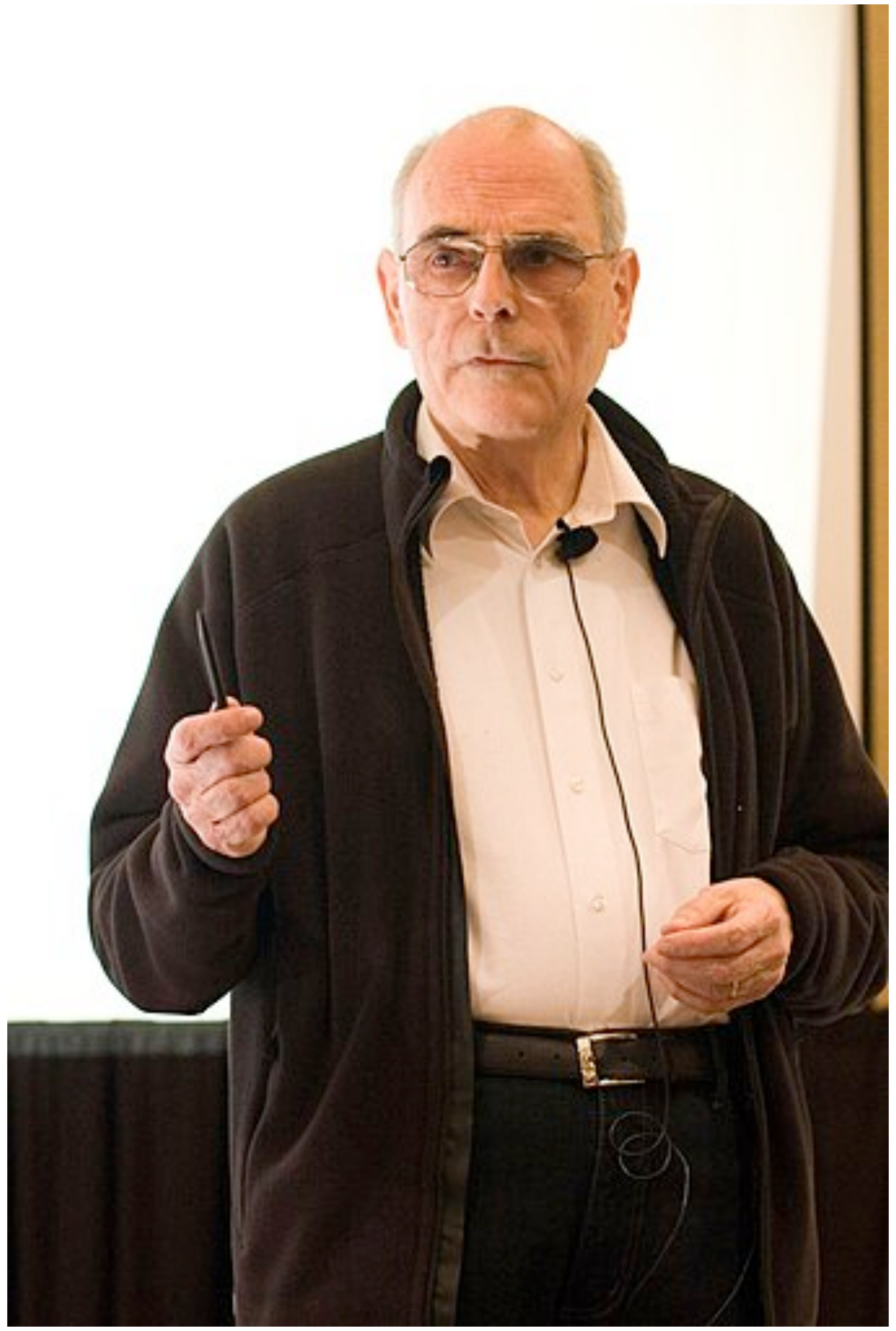
Strassen's algorithm for Matrix Multiplication

- $X = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ and $Y = \begin{bmatrix} E & F \\ G & H \end{bmatrix}$
- $XY = \begin{bmatrix} P_5 + P_4 - P_2 + P_6 & P_1 + P_2 \\ P_3 + P_4 & P_1 + P_5 - P_3 - P_7 \end{bmatrix}$
  - where:

$$P_1 = A(F - H),\ P_2 = (A + B)H,\ P_3 = (C + D)E,\ P_4 = D(G - E)$$

$$P_5 = (A + D)(E + H),\ P_6 = (B - D)(G + H),\ P_7 = (A - C)(E + F)$$

- Recurrence: $T(n) = 7 \cdot T(n/2) + \Theta(n^2)$



Invented by Volker Strassen at 1969

Time complexity of Strassen's algorithm

- The **substitution** method (or, **guess and verify**)
  - Guess the form of the solution;
  - Use induction to find proper constants and prove the solution works

Time complexity of Strassen's algorithm

- Recurrence: $T(n) = 7 \cdot T(n/2) + \Theta(n^2)$
- $T(n) = 7 \cdot T(n/2) + cn^2, T(1) = c$
- Let's guess $T(n) \leq d \cdot n^{\log_2 7} = O(n^{log_2 7})$
- **Induction basis**:
  - $T(1) = c \leq d \cdot 1^{\log_2 7}$, as long as $d \geq c$
- **Inductive step**:
  - $T(n) = 7 \cdot T(n/2) + cn^2 \leq 7d(n/2)^{\log_2 7} + cn^2 = dn^{\log_2 7} + cn^2$

Inconsistant!

Time complexity of Strassen's algorithm

- $T(n) = 7 \cdot T(n/2) + cn^2, T(1) = c$
- The guess $T(n) \leq d \cdot n^{\log_2 7} = O(n^{log_2 7})$ does not work out…
- However, in fact, $O(n^{\log_2 7})$ is the right answer…
  - So we add some lower order term (such as n^2) to our guess?
  - No, we should **subtract** some lower order term from our guess!
  - **Subtraction** gives us stronger induction hypothesis to work with!

Time complexity of Strassen's algorithm

- $T(n) = 7 \cdot T(n/2) + cn^2, T(1) = c$
- Guess $T(n) \le dn^{\log_2 7} - d'n^2 = O(n^{\log_2 7})$
- **Induction basis**:
  - $T(1) = c \le d \cdot 1^{\log_2 7} - d' \cdot 1^2$, as long as $d - d' \ge c$
- **Inductive step**:
  - $T(n) = 7 \cdot T(n/2) + cn^2 \le 7d(n/2)^{\log_2 7} - 7d'(n/2)^{\log_2 7} + cn^2$

    $= dn^{\log_2 7} - (7d'/4 - c)n^2 \le dn^{\log_2 7} - d'n^2$, as long as $3d'/4 \ge c$

Making a good guess

- There is no general way to correctly guess the tightest asymptotic solution to an arbitrary recurrence.
- Making a good guess takes experience and, occasionally, creativity.
- Sometimes need to repeat the guessing process (first determine loose upper and lower bounds on the recurrence and then reduce your range of uncertainty)

Further reading

- [CLRS] Ch.2 (2.3), Ch.4
- [Erickson] Ch.1 (excluding 1.5 and 1.8)