贪心策略
Greedy Strategy

钮鑫涛
Nanjing University
2024 Fall

The slides are mainly adapted from the original ones shared by Chaodong Zheng and Kevin Wayne. Thanks for their supports!

The Greedy Strategy

- For many games, you should think ahead, a strategy which focuses on immediate advantage could easily lead to defeat.
  - Such as playing chess.
- But for many other games, you can do quite well by simply making whichever move seems best at the moment, without worrying too much about future consequences.
  - Such as building an MST.

The Greedy Strategy

- **The Greedy Algorithmic Strategy**: given a problem, build up a solution piece by piece, always choosing the next piece that offers the most obvious and immediate benefit.
  - Sometimes it gives optimal solution.
  - Sometimes it gives near-optimal solution.
  - Or, it simply fails…

An Activity-Selection Problem

- Assume we have one hall and n activities $S = \{a_1, \cdots, a_n\}$.
  - Each activity has a start time s_i and a finish time f_i.
  - Two activities cannot happen simultaneously in the hall.
  - Maximum number of activities we can schedule?

An Activity-Selection Problem

- Let's start with "divide-and-conquer"
  - Define S_i to be the set of activities start after a_i finishes;
  - Define F_i to be the set of activities finish before a_i starts.
  - $OPT(S) = \max_{1 \le i \le n} \{OPT(F_i) + 1 + OPT(S_i)\}$

In any solution, some activity is the first to finish.

$$OPT(S) = \max_{1 \le i \le n} \{1 + OPT(S_i)\}$$

Observation: To make $\text{OPT}(S)$ as large as possible, the activity that finishes first should finish as early as possible!

An Activity-Selection Problem

- A greedy strategy to solve this problem:

ActivitySelection(S):

Sort S into increasing order of finish time

$SOL := \{a_1\},\ a' = a_1$

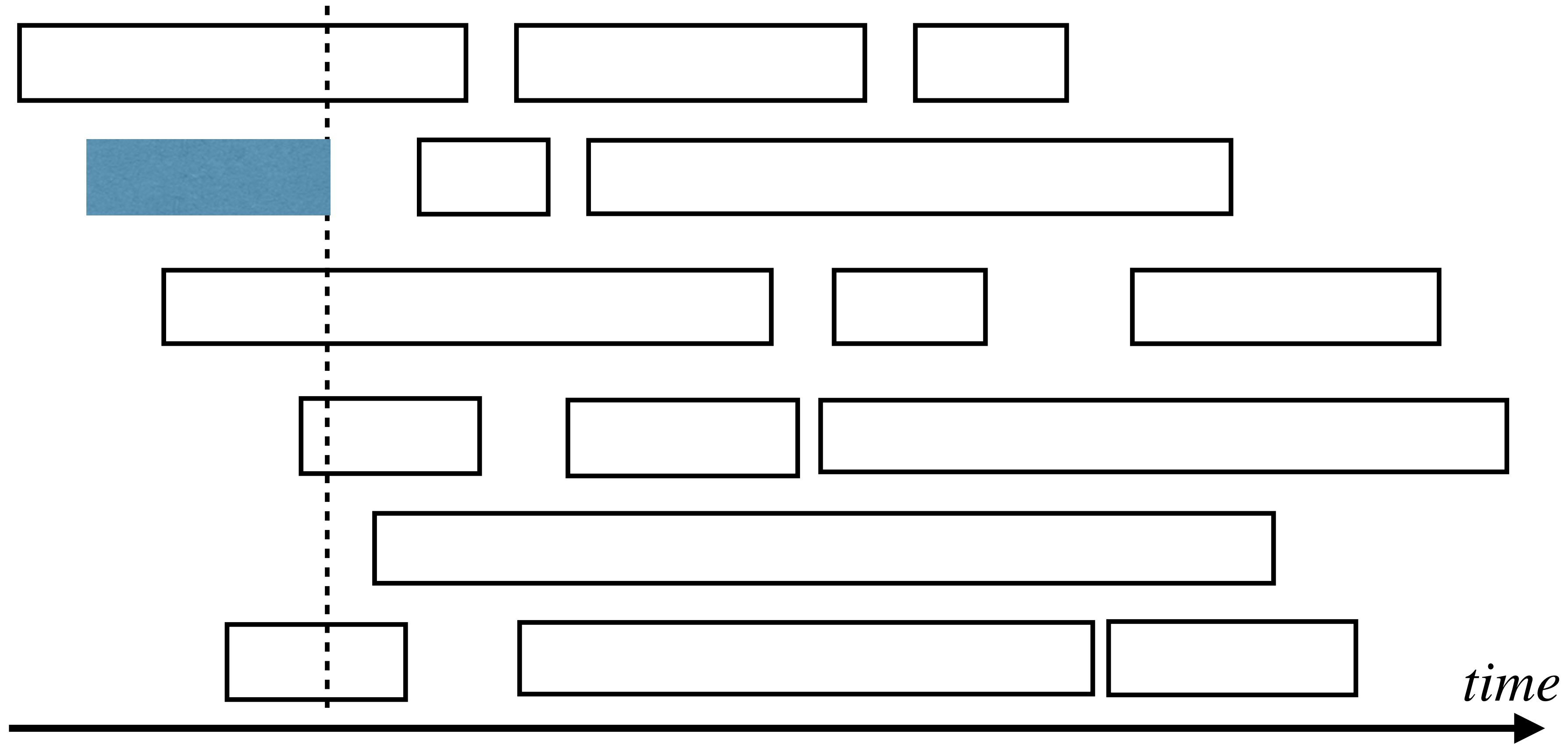
for $i := 2$ **to** n

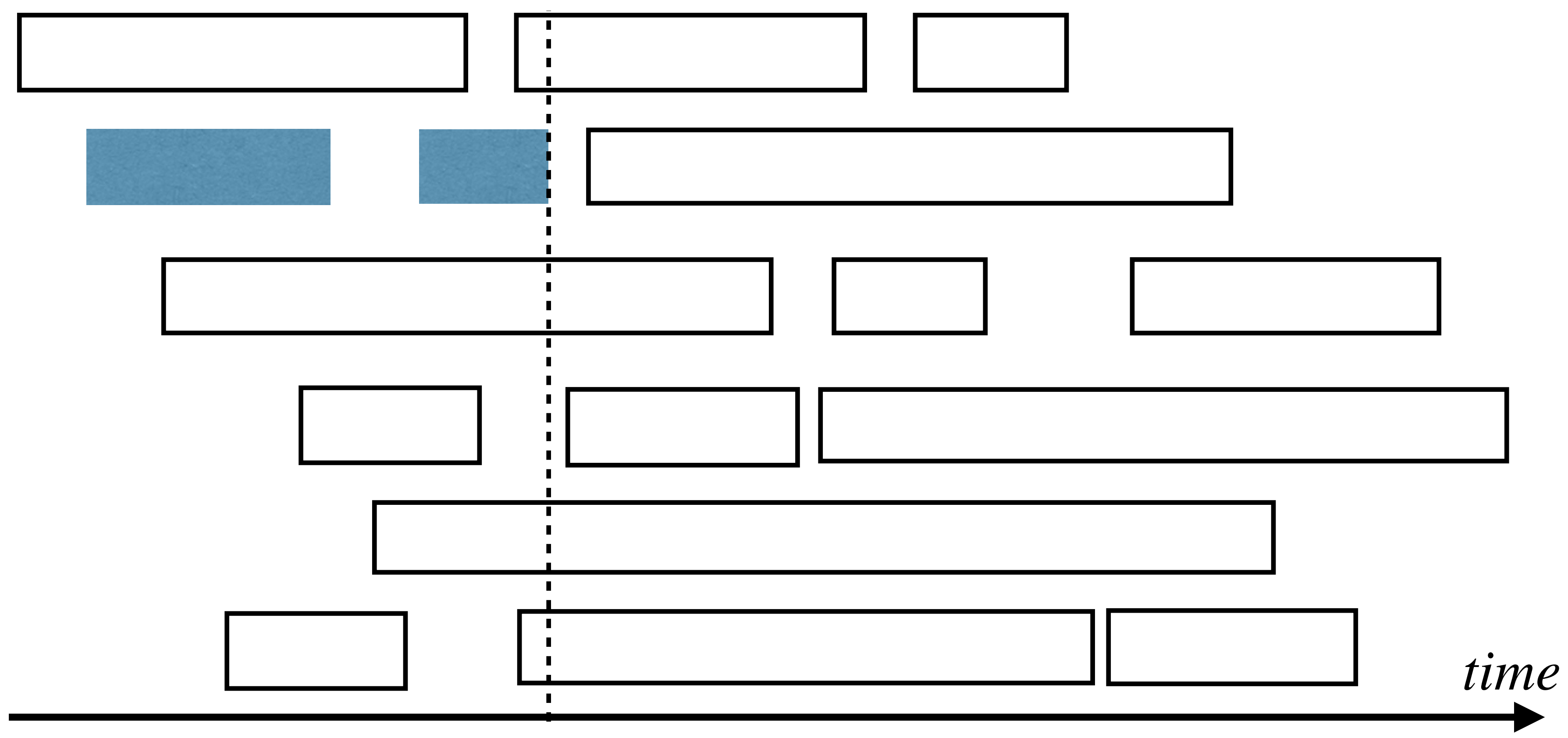
  If $a_i.start\_time > a'.finish\_time$

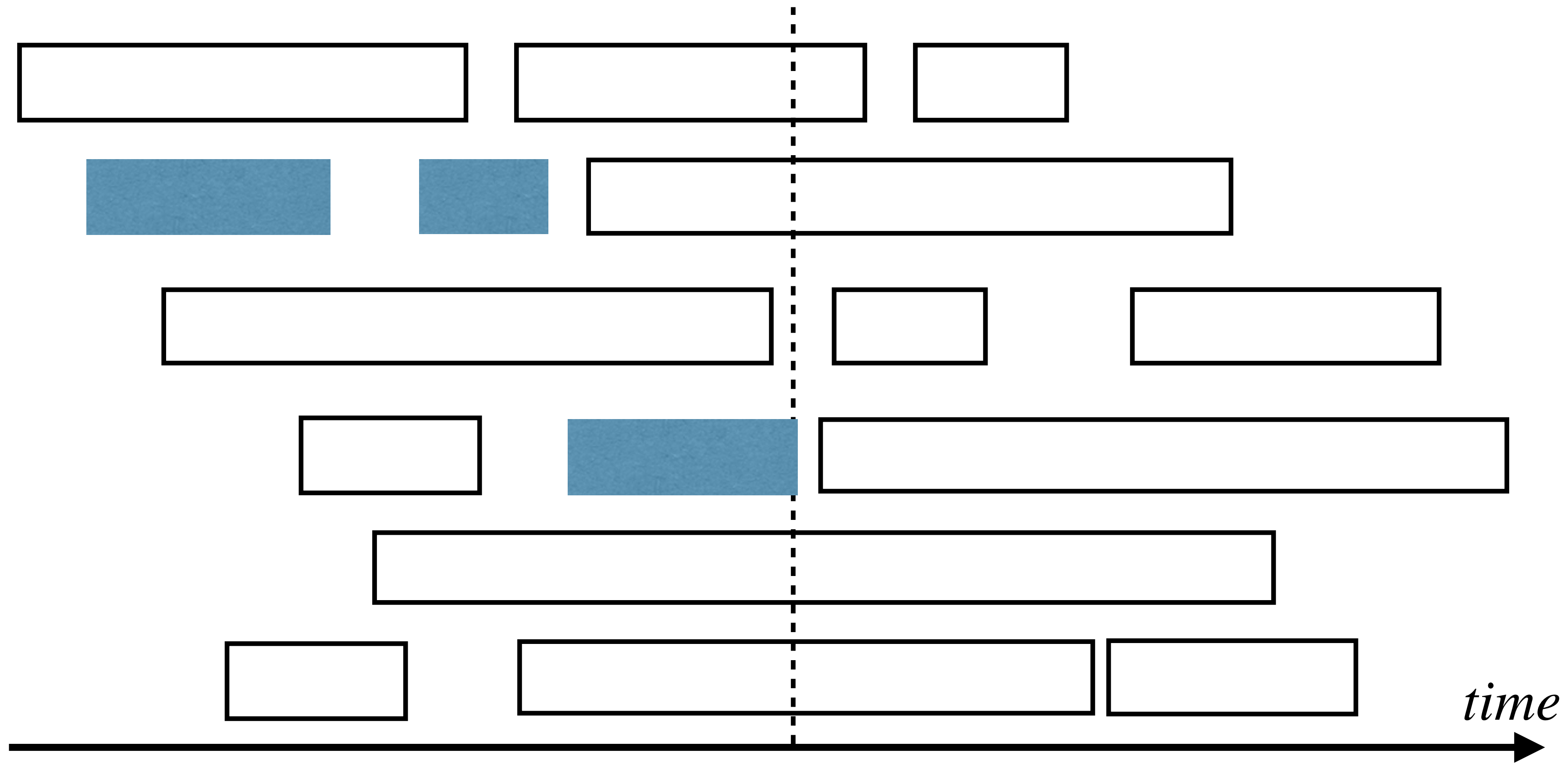
    $SOL := SOL \cup \{a_i\}$

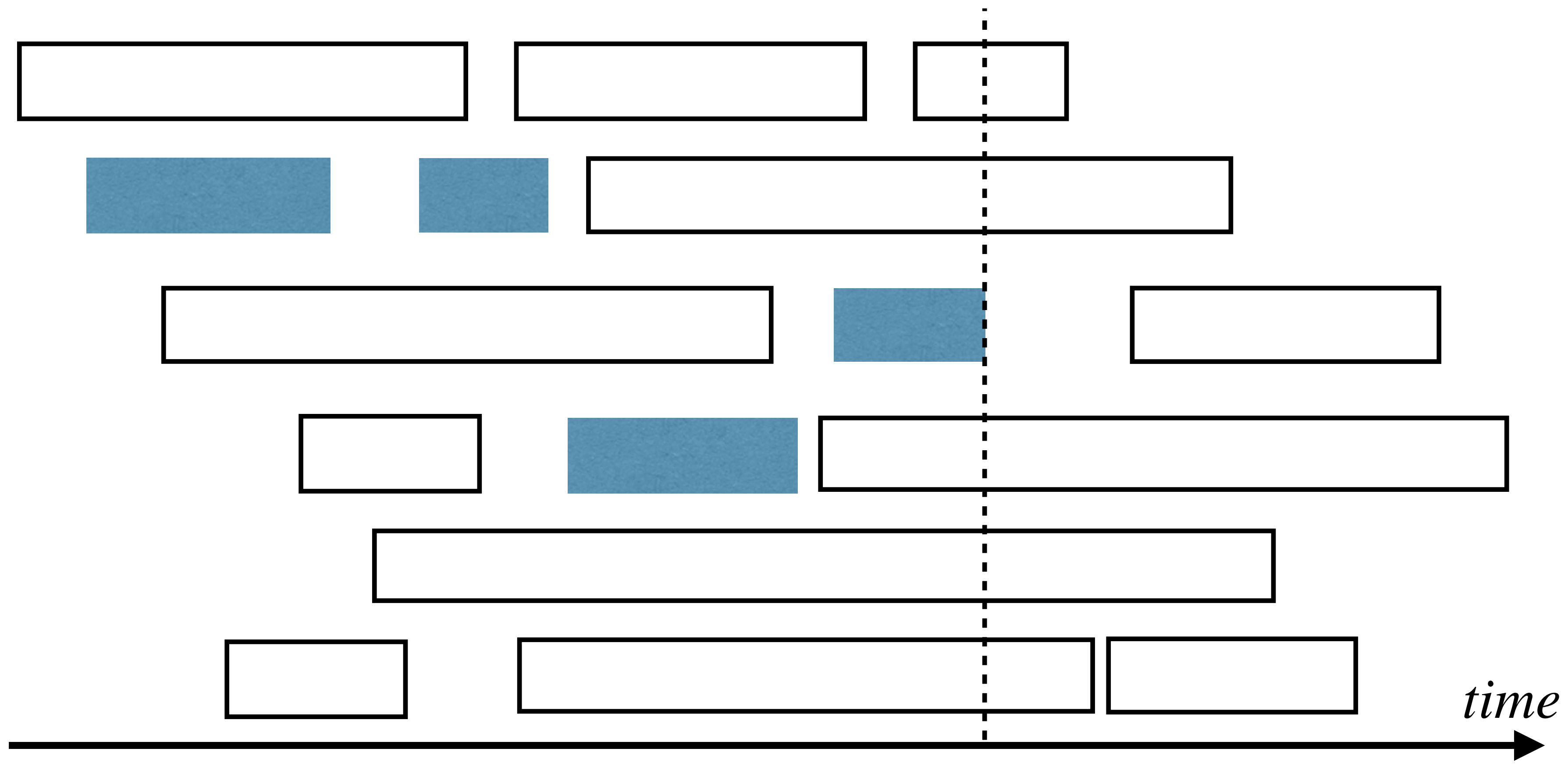
    $a' := a_i$

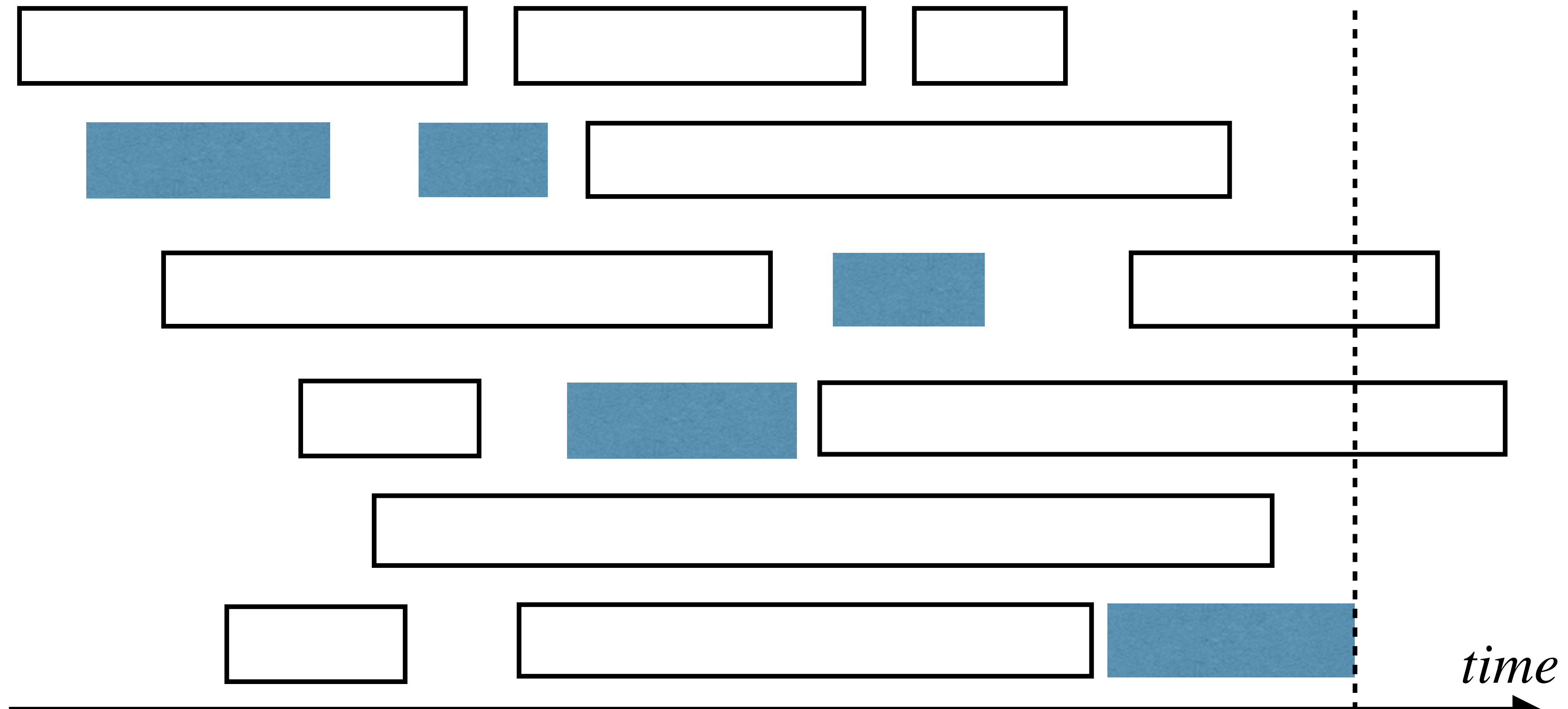
return SOL


time


time


time


time


time

Correctness of the greedy strategy for this problem

- The Greedy Algorithm for the Activity-Selection Problem:
  - Add earliest finish activity a' to solution, remove ones overlapping with a'.
  - Repeat until all activities are processed.
- How to formally prove this algorithm is correct?
  - The firstly selected activity is in some optimal solution.
  - The following selection is correct to this optimal solution.

Correctness of the greedy strategy for this problem

Lemma 1 let a' be the earliest finishing activity in S, then a' is in some optimal solution of the problem.

- Proof:
  - Let $OPT(S)$ be an optimal solution to the problem, let a be the earliest finishing activity in $OPT(S)$.
  - Assume $a' \notin OPT(S)$, otherwise we are done.
  - Then $SOL(S) = OPT(S) + a' - a$ is also a feasible solution, and it has same size as $OPT(S)$.
  - So $SOL(S)$ is also an optimal solution.

Correctness of the greedy strategy for this problem

Lemma 2 let a' be the earliest finishing activity in S, let S' be the activities starting after a', then $OPT(S') \cup \{a'\}$ is an optimal solution of the problem.

- Proof:
  - Let $OPT(S)$ be an optimal solution to the original problem, and $a' \in OPT(S)$. (Lemma 1 ensures such solution exists.)
  - Thus, $OPT(S) = SOL(S') \cup \{a'\}$.
  - If $OPT(S') \cup \{a'\}$ is not an optimal solution to the original problem, then it must be the case that $|SOL(S')| > |OPT(S')|$.
  - But this contradicts that $OPT(S')$ is an optimal solution for problem S'.

Correctness of the greedy strategy for this problem

Theorem The greedy algorithm for the activity-selection problem is correct.

- Proof:
  - By induction on size of S.
  - When $|S| = 1$, the algorithm clearly is correct.
  - When $|S| = n$. Due to Lemma 2, $OPT(S) = OPT(S') \cup \{a'\}$
  - By induction hypothesis, the algorithm correctly finds $OPT(S')$. So we are done.

Elements of the Greedy Strategy

Elements of the Greedy Strategy

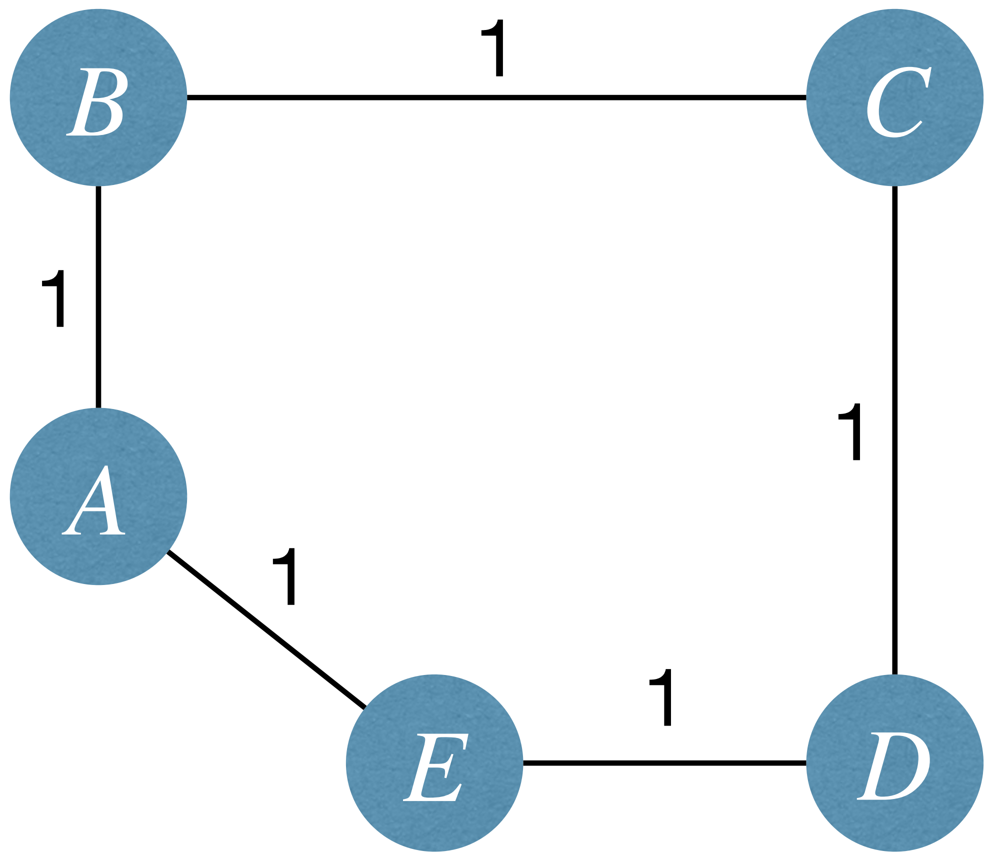
- If an (optimization) problem has following two properties, then the greedy strategy usually works for it:
    - **Optimal substructure.**
    - **Greedy property**.

Optimal Substructure

- A problem exhibits optimal substructure if an optimal solution to the problem contains within it optimal solution(s) to subproblem(s):
  - Size n problem $P(n)$, and optimal solution of $P(n)$ is $OPT_{P(n)}$.
  - Solving $P(n)$ needs to solve size $n' < n$ subproblem $P(n')$.
  - Optimal solution of $P(n')$: $OPT_{P(n')}$
  - $OPT_{P(n)}$ contains a solution of $P(n')$: $SOL_{P(n')}$
  - Optimal Substructure Property: $SOL_{P(n')}$ = $OPT_{P(n')}$
    - Or these two solutions provide same “utility” under certain metric.

Optimal Substructure

- Example:
  - Lemma 2 in activity selection: let a' be the earliest finishing activity in S, let S' be the activities starting after a', then $OPT(S') \cup \{a'\}$ is some $OPT(S)$.
- There are problems that do **NOT** exhibit optimal substructure property!
  - E.g., find the longest path between two vertices without repeating an edge.



Greedy-Choice Property

- At each step when building a solution, make the choice that looks best for the *current* problem, *without* considering results from subproblems. That is, make local greedy choice at each step.

  - To solve $P(n)$, currently have k choices a_1 to a_k. If we choose a_i, the problem is reduced to a smaller size n_i subproblem $P(n_i)$.

  - If the problem only admits optimal structure:

    - Find i that maximize, Utility$(a_i + OPT_{P(n_i)})$.

    - We have to compute $OPT_{P(n_i)}$ for all i first.

Greedy-Choice Property

Identifying a greedy-choice property is the challenging part!

- With greedy choice:
  - We **have a way** to pick correct i, without knowing any $OPT_{P(n_i)}$.
- Example:
  - Lemma 1 in activity selection: let a' be the earliest finishing activity in S, then a' is in some optimal solution of the problem.

Fractional Knapsack Problem

- A thief robbing a warehouse finds n items $A = \{a_1, \ldots, a_n\}$.
- Item a_i is worth v_i dollars and weighs w_i pounds.
- The thief can carry at most W pounds in his knapsack.
- The thief can carry **fraction of items.**
- What should the thief take to maximize his profit?

Fractional Knapsack Problem

- A greedy strategy:
  - keep taking the most cost efficient item (i.e., $\max\{\frac{v_i}{w_i}\}$) until the knapsack is full.
- The greedy solution is optimal!
  - Greedy-choice
  - Optimal substructure

Correctness of the greedy algorithm

- Lemma 1 [greedy-choice]: let a_m be a most cost efficient item in A, then in some optimal solution, at least $w_{m'} = \min\{w_m, W\}$ pounds of a_m are taken.
- Proof:
  - Consider an optimal solution, assume $w' < w_{m'}$ pounds of a_m are taken.
  - Now, substitute $w_{m'} - w'$ pounds of other items with a_m.
  - Since a_m is most cost-efficient, the new solution cannot be worse.

Correctness of the greedy algorithm

- Lemma 2 [optimal substructure]: let a_m be a most cost efficient item in A, then "$OPT_{W-\min\{w_m,W\}}(A-a_m)$ with $\min\{w_m, W\}$ pounds of a_m" is an optimal solution of the problem.
- Proof:
    - Consider some $OPT_{W(A)}$ containing $\min\{w_m, W\}$ pounds of a_m.
    - If optimal substructure does not hold, then $OPT_{W(A)}$ gives $SOL_{W-\min\{w_m,W\}}(A-a_m) > OPT_{W-\min\{w_m,W\}}(A-a_m)$.
    - But this contradicts the optimality of $OPT_{W-\min\{w_m,W\}}(A-a_m)$.

0-1 Knapsack Problem

- A thief robbing a warehouse finds n items $A = \{a_1, \ldots, a_n\}$.
- Item a_i is worth v_i dollars and weighs w_i pounds.
- The thief can carry at most W pounds in his knapsack.
- The thief ***cannot*** carry fraction of items!
- What should the thief take to maximize his profit?

0-1 Knapsack Problem

- A greedy strategy:
  - keep taking the most cost efficient item (i.e., $\max\{\frac{v_i}{w_i}\}$) until the knapsack is full.
- The greedy solution is **NOT** optimal!
- A simple **counterexample**:
  - There are only two items.
  - Item One has value 2 and weighs 1 pound.
  - Item Two has value W and weighs W pounds.

The greedy solution can be arbitrarily bad!

Why greedy strategy fail?

- Lemma 1 [greedy-choice]: let a_m be a most cost efficient item that can fit into the bag, then in some optimal solution, this item is taken.

- Thus, this lemma cannot be proven!
  - Consider an optimal solution, assume a_m is NOT taken.
  - Now, substitute $w' \geq w_m$ pounds of other items with a_m (all w_m pounds).
  - However, these w' pounds of items may have aggregate value larger than v_m, since it may $w' > w_m$.

can $w' < w_m$?

What about the optimal substructure property? That is, is $OPT_{W-w_x}(A - a_x)$ with w_x pounds of a_x is the optimal solution?

A data compression problem

- Assume we have a data file containing 100k characters.
  - Further assume the file only uses 6 characters.
  - How to store this file to save space?
- Simplest way: use 3 bits to encode each char.
  - `a=000,b=001,…,f=101`
  - This costs 300k **bits** in total.
- Can we do better?

A data compression problem

- How to store this file to save space?
  - Instead of using fixed-length codeword for each char, we should **let frequent chars use shorter codewords.** That is, use a *variable-length code.*

	a	b	c	d	e	f
Frequency	45k	13k	12k	16k	9k	5k
Fixed-length code	000	001	010	011	100	101
varaible-length code	0	00	01	1	10	11

How to decode bit string `000`?

A data compression problem

- How to store this file to save space?
  - Instead of using fixed-length codeword for each char, we should **let frequent chars use shorter codewords.** That is, use a *variable-length code.*
  - To avoid ambiguity in decoding, variable-length code should be **prefix-free**: **no codeword is also a prefix** of some other codeword.

	a	b	c	d	e	f
Frequency	45k	13k	12k	16k	9k	5k
Fixed-length code	000	001	010	011	100	101
varaible-length code	0	101	100	111	1101	1100

Fixed-length code vs Variable-length code: 300k vs 224k. This is ≈25% saving.

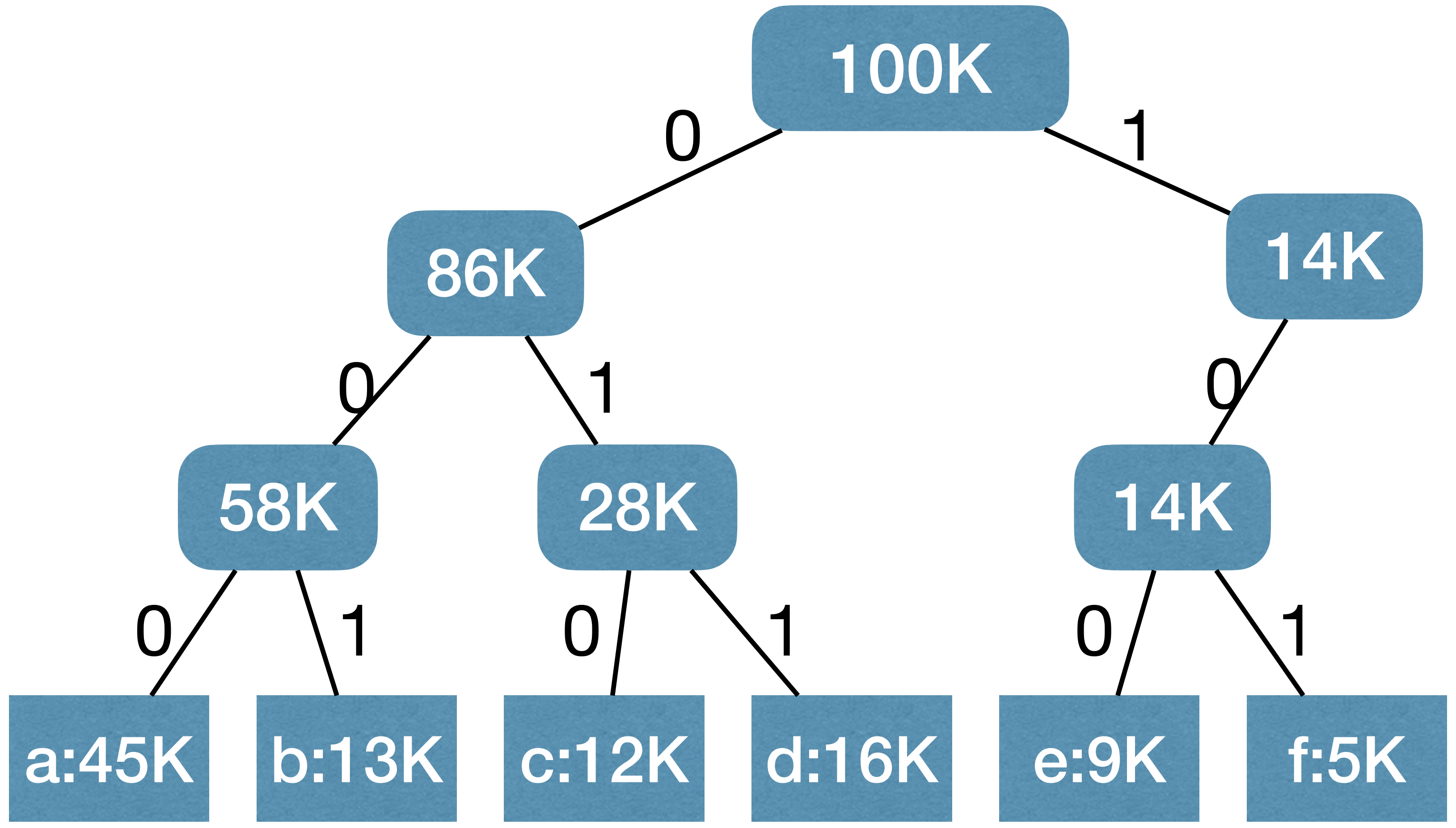
Is it optimal?

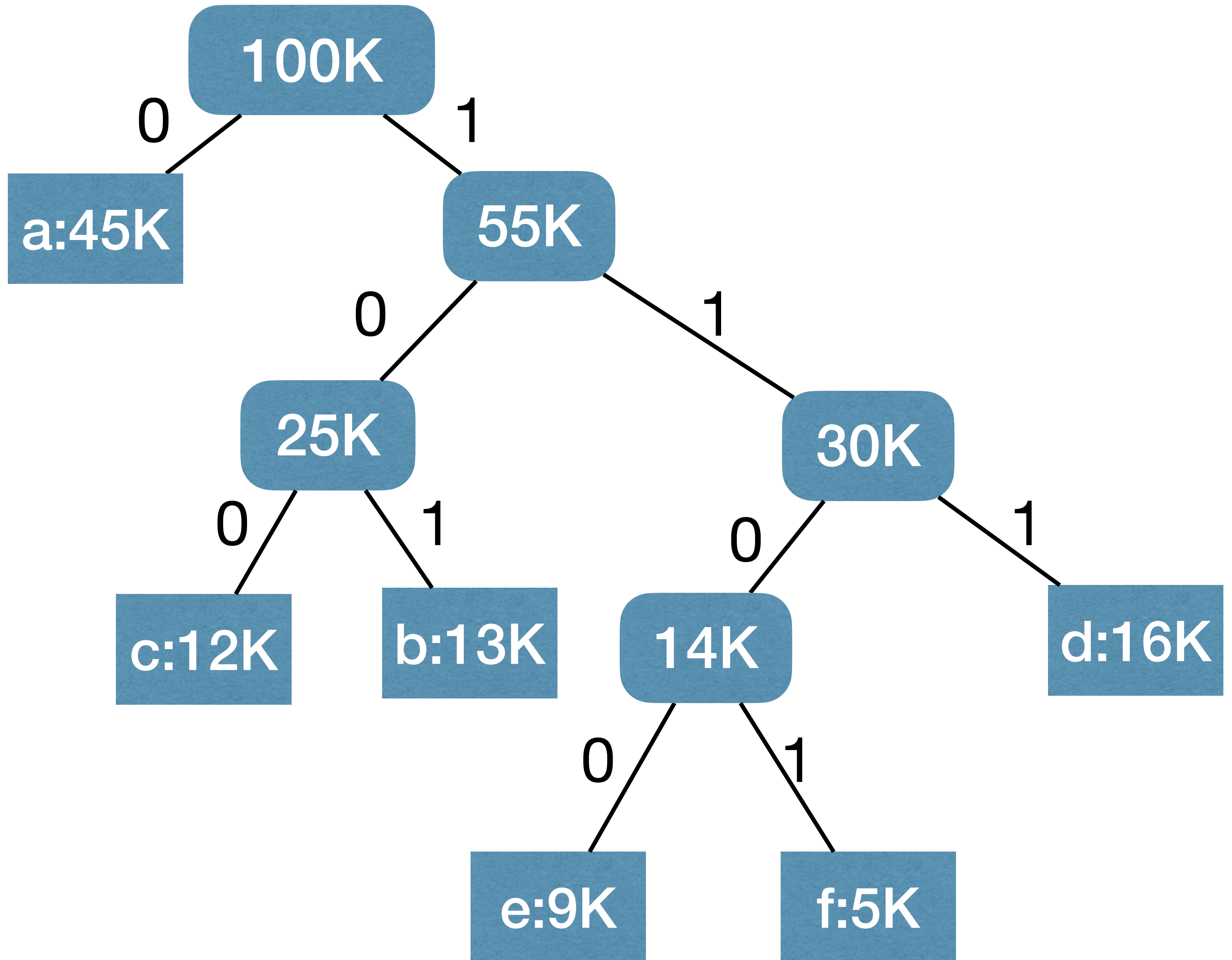
Properties of prefix-free code

- Use a binary tree to visualize a prefix-free code.
  - Each leaf denotes a char.
  - Each internal node: left branch is 0, right branch is 1.
  - Path from root to leaf is the codeword of that char.
  - **Optimal code must be represented by a *full binary tree*: a tree each node having zero or two children.**

WHY?

	a	b	c	d	e	f
Frequency	45k	13k	12k	16k	9k	5k
Fixed-length code	000	001	010	011	100	101
varaible-length code	0	101	100	111	1101	1100





Length of encoded message

- Consider a file using a size n alphabet $C = \{c_1, \ldots, c_n\}$. For each character, let f_i be the frequency of char c_i.

- Let T be a full binary tree representing a prefix-free code.For each character c_i, let $d_T(i)$ be the depth of c_i in T.

  - Length of encoded message is $\sum_{i=1}^{n} f_i \cdot d_T(i)$

- Alternatively, recursively (bottom-up) define each internal node's frequency to be sum of its two children.

  - Length of encoded message is $\sum_{u \in tree \setminus root} f_u$

Huffman Codes

- How to construct optimal prefix-free code?
- Huffman Codes: Merge the two least frequent chars and recurse.

Huffman(C):

Build a priority queue Q based on frequency

for $i := 1$ **to** $n - 1$

   Allocate new node z

   $x := z.left := Q.ExtractMin()$

   $y := z.right := Q.ExtractMin()$

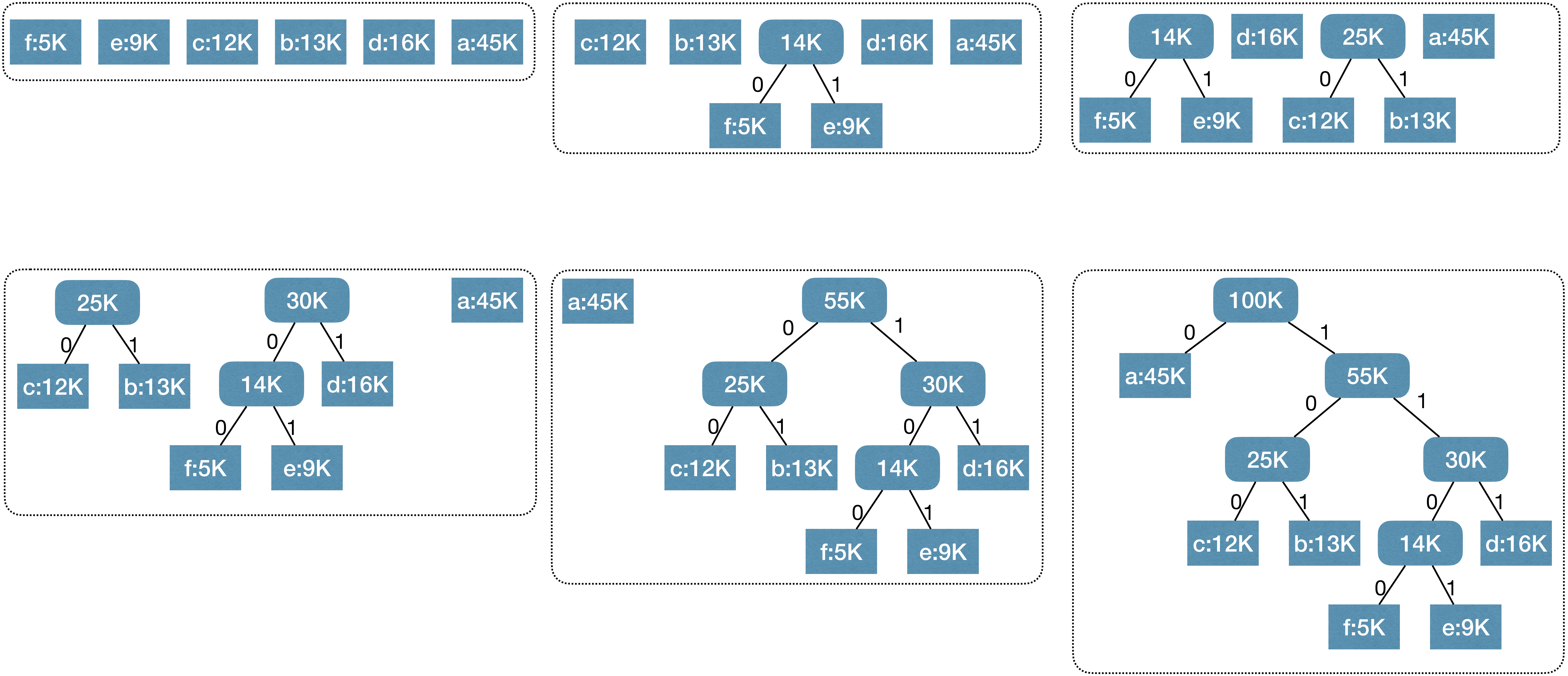
   $z.frequency := x.frequency + y.frequency$

   $Q.Insert(z)$

return $Q.ExtractMin()$

→ Time complexity is $O(n \log n)$

Huffman Codes



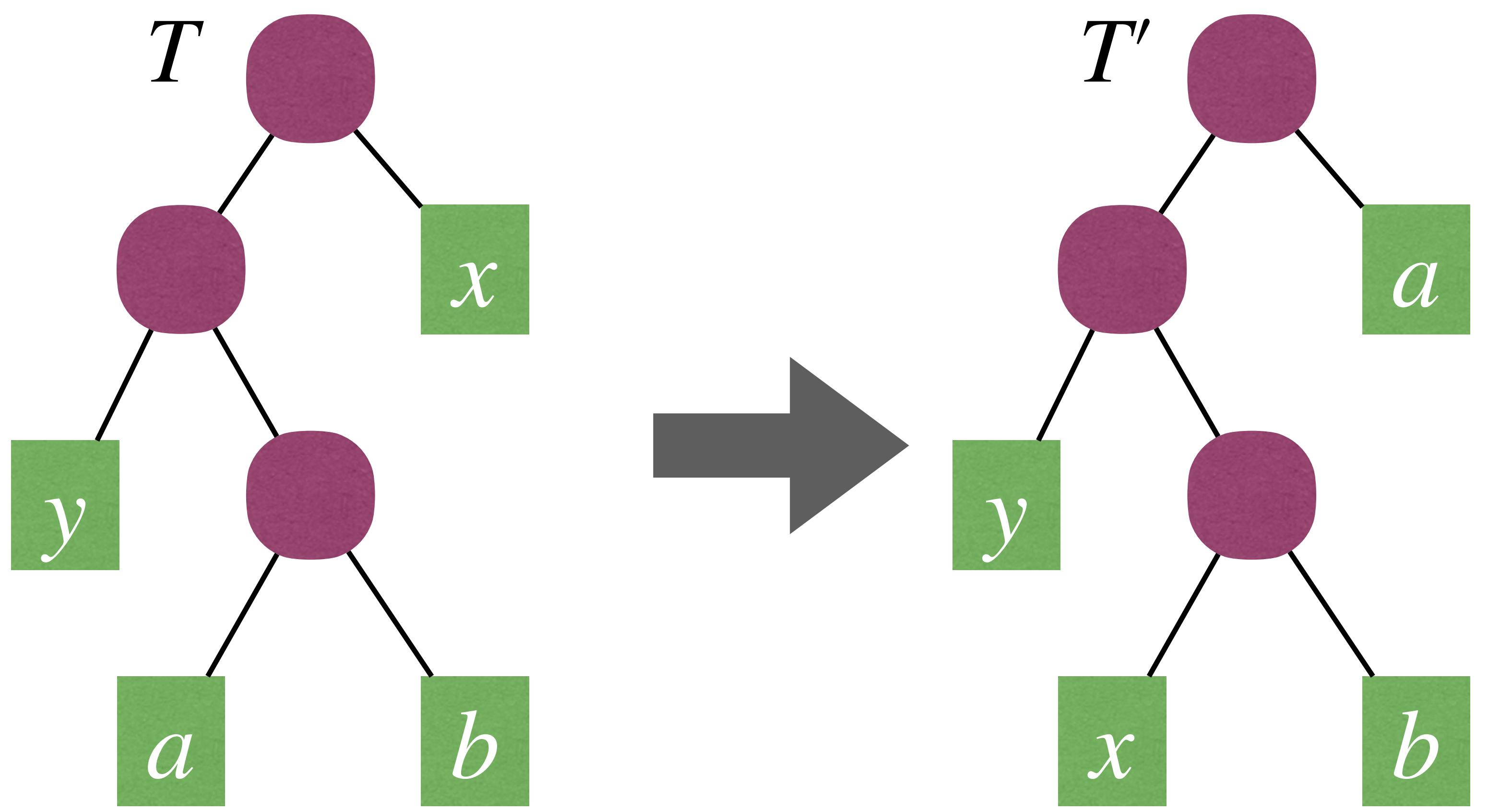
Correctness of Huffman Codes

- Length of encoded message is computed by $\sum_{i=1}^{n} f_i \cdot d_T(i)$ or $\sum_{u \in tree \setminus root} f_u$
- Huffman Codes: Merge the two least frequent chars and recurse.
- Lemma 1 [greedy choice]: Let x and y be two least frequent chars, then in some optimal code tree, x and y are siblings and have largest depth.
- Lemma 2 [optimal substructure]: Let x and y be two least frequent chars in C. Let $C_z = C - \{x, y\} + \{z\}$ with $f_z = f_x + f_y$. Let T_z be an optimal code tree for C_z. Let T be a code tree obtained from T_z by replacing leaf node z with an internal node having x and y as children. Then, T is an optimal code tree for C.

Correctness of Huffman Codes

Lemma 1 [greedy choice]: Let x and y be two least frequent chars, then in some optimal code tree, x and y are siblings and have largest depth.

- Proof sketch:
  - Let T be an optimal code tree with depth d.
  - Let a and b be siblings with depth d. (Recall T is a full binary tree.)
  - Assume a and b are **not** x and y. (Otherwise we are done.)
  - Let T' be the code tree obtained by swapping a and x.
  - $cost(T') = cost(T) + (d - d_T(x)) \cdot f_x - (d - d_T(x)) \cdot f_a = cost(T) + (d - d_T(x)) \cdot (f_x - f_a) \leq cost(T)$
  - Swapping b and y, obtaining T'', further reduces the total cost.
  - So T'' must also be an optimal code tree.



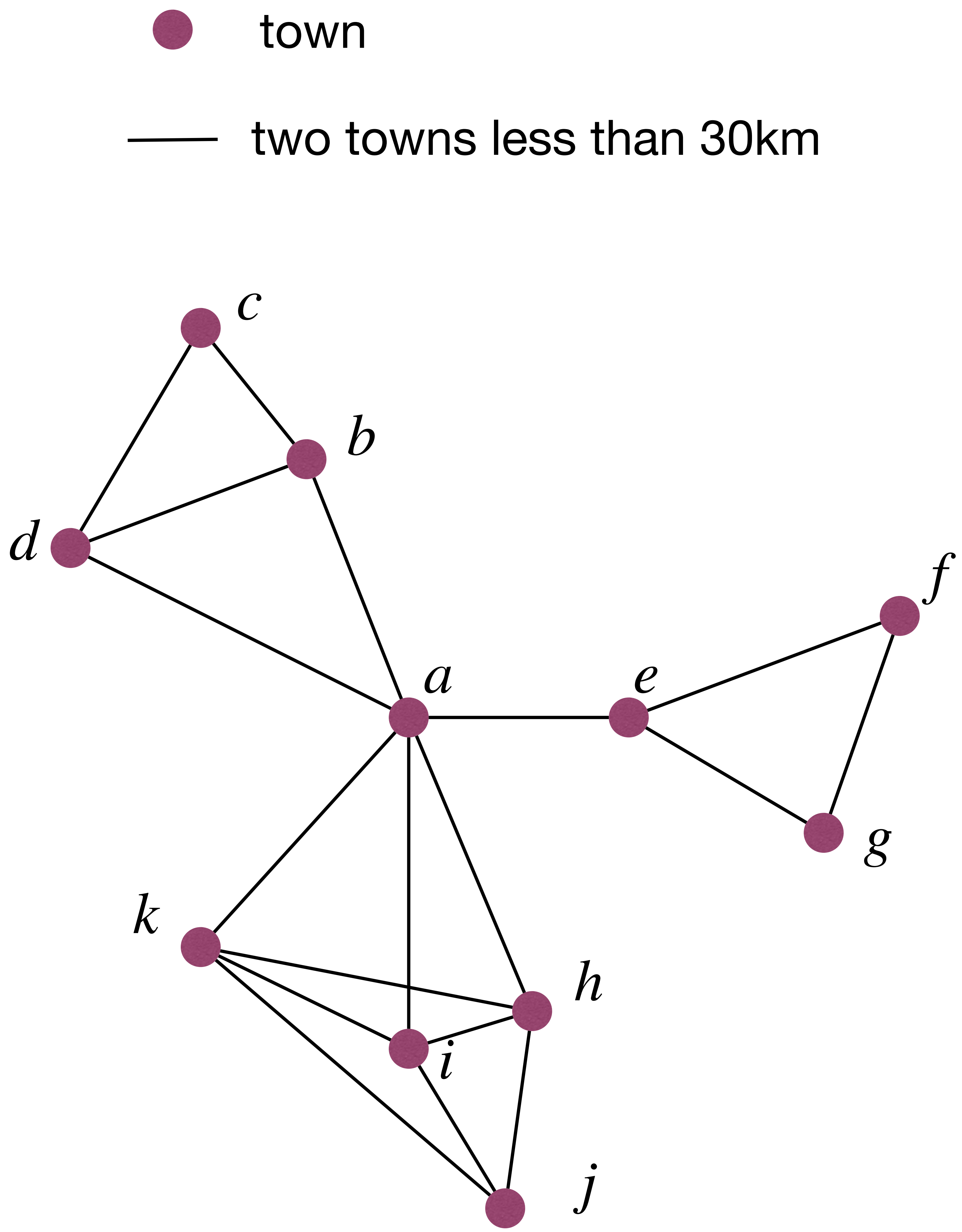
Correctness of Huffman Codes

Lemma 2 [optimal substructure]: Let x and y be two least frequent chars in C. Let $C_z = C - \{x, y\} + \{z\}$ with $f_z = f_x + f_y$. Let T_z be an optimal code tree for C_z. Let T be a code tree obtained from T_z by replacing leaf node z with an internal node having x and y as children. Then, T is an optimal code tree for C.

- Proof sketch:
- Let T' be an optimal code tree for C, with x and y being sibling leaves.
- $Cost(T') = f_x + f_y + \sum_{u \in T' \setminus root \text{ and } u \notin \{x,y\}} f_u = f_x + f_y + cost(T'_z) \geq f_x + f_y + cost(T_z) = cost(T)$
- So T must be an optimal code tree for C.

Set Cover

- Suppose we need to build schools for n towns.
- Each school must be in a town, no child should travel more than 30km to reach a school.
- Minimum number of schools we need to build?



Set Cover

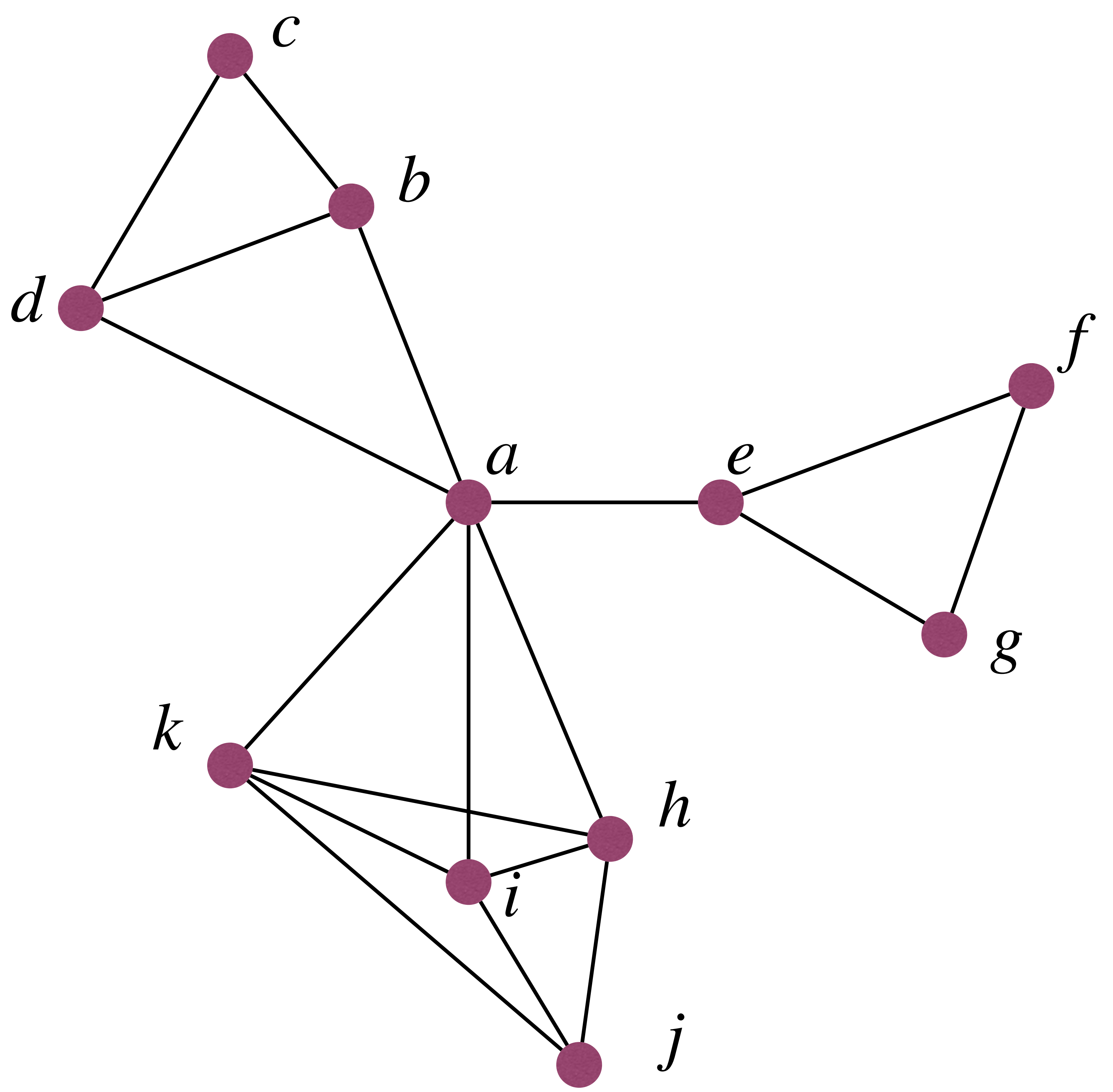
- The Set Cover Problem:

- **Input:** a universe U of n elements; and $\mathcal{S} = \{S_1, \ldots, S_m\}$ where each $S_i \subseteq U$.

- **Output:** $\mathscr{C} \subseteq \mathcal{S}$ such that $\bigcup_{S_i \in \mathscr{C}} S_i = U$

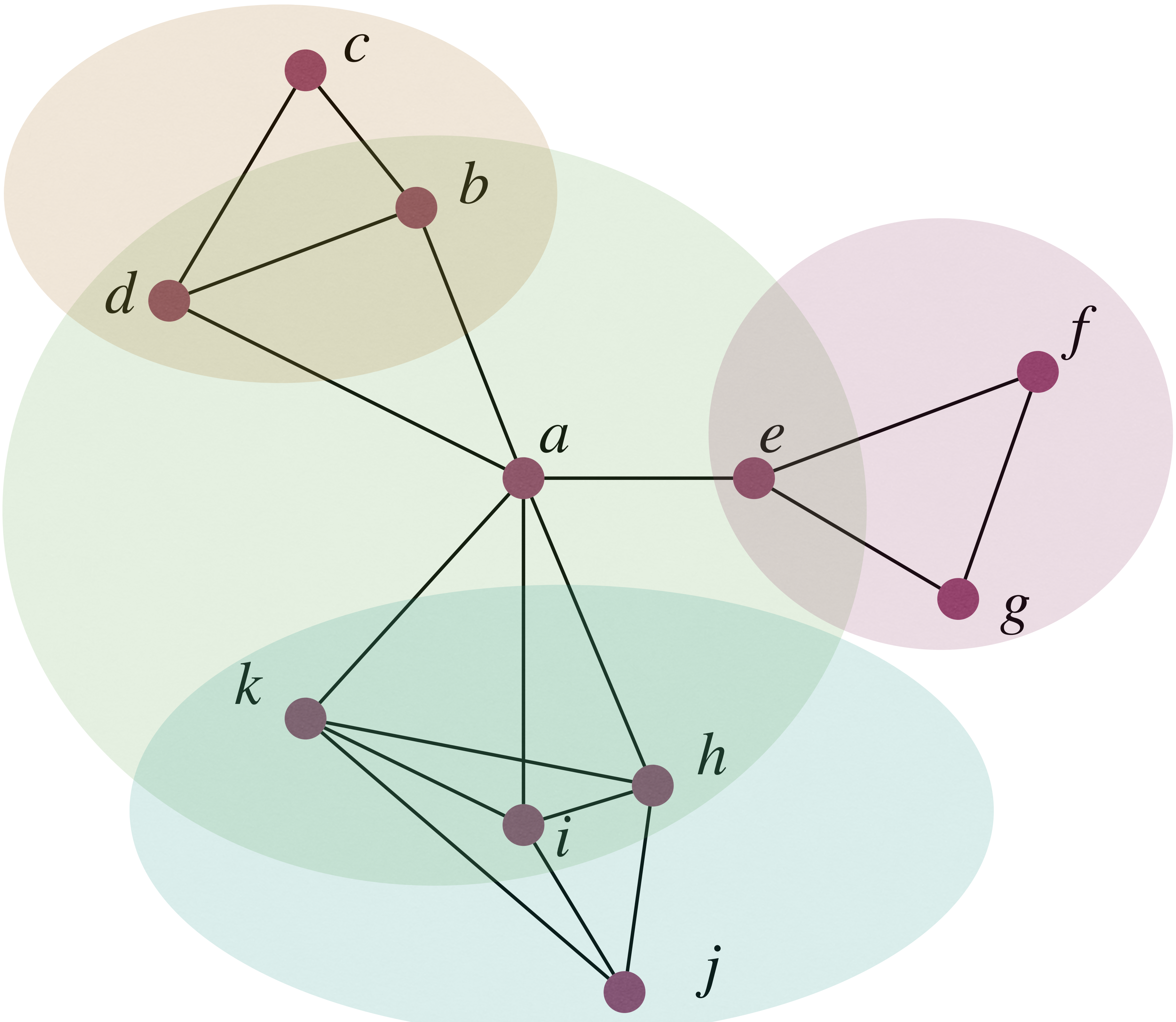
  - That is, a subset of $\mathcal{S}$ that "covers" U.

- **Goal:** minimize $|\mathscr{C}|$



Set Cover

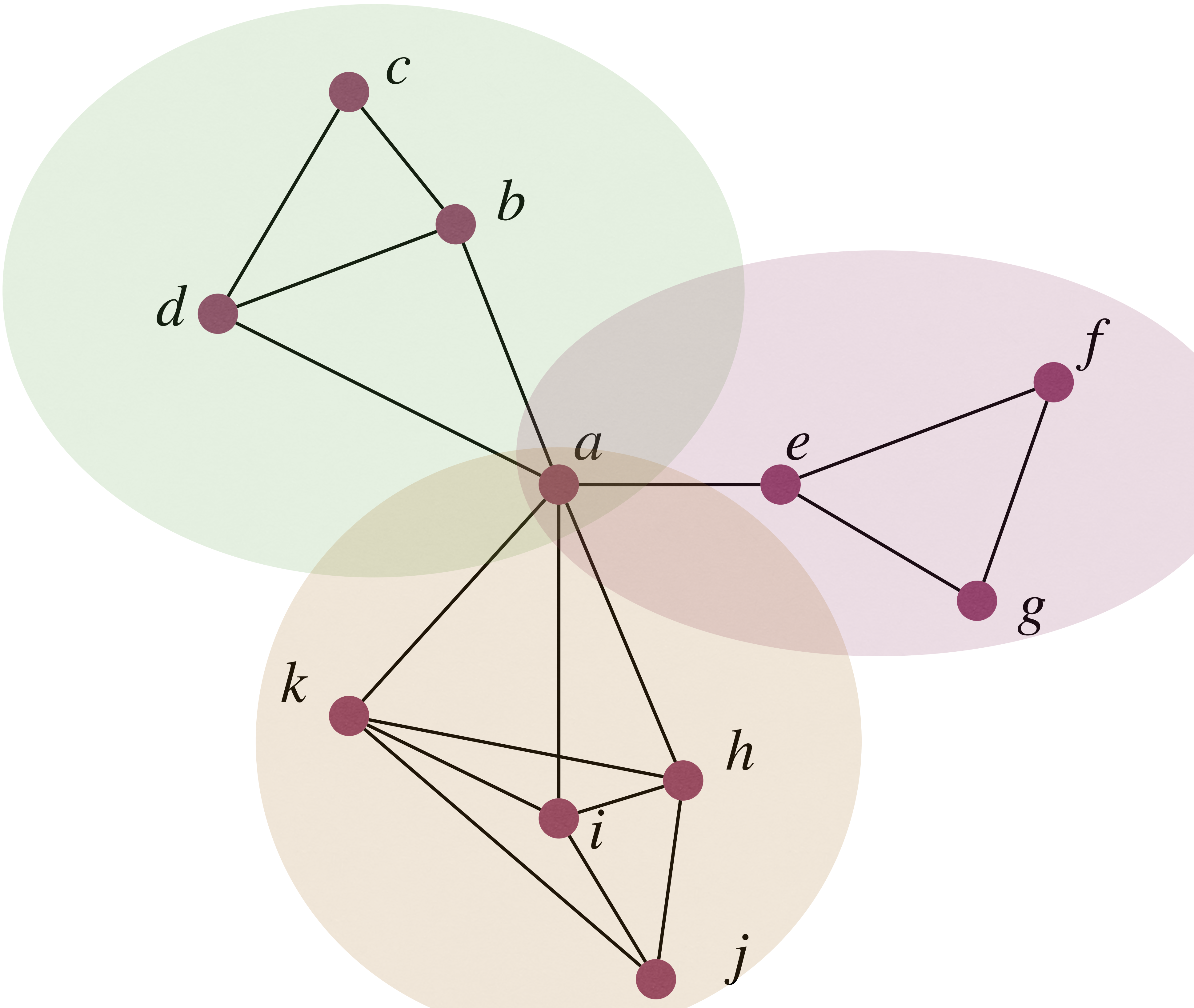
- **Simple greedy strategy**:
- Keep picking the town that covers most remaining uncovered towns, until we are done.
  - Pick the set that covers most uncovered elements, until all elements are covered.
- Greedy solution: a, f, c, j



Can we do better?

Set Cover

- The optimal solution is b, e, i
- Nevertheless, the greedy solution a, f, c, j is very close!
  - But, how close?



Greedy solution of Set Cover is close to optimal

Theorem Suppose the optimal solution uses k sets, then the greedy strategy will use at most $k \ln n$ sets.

- **Proof:**
- Let n_t be number of uncovered elements after t iterations. (Thus $n_0 = n$.)
- These n_t elements can be covered by some k sets. (The optimal solution will do)
- So one of the remaining sets will cover at least $\frac{n_t}{k}$ of these uncovered elements.
- Thus $n_{t+1} \leq n_t - \frac{n_t}{k} = n_t(1 - \frac{1}{k})$
- $n_t \leq n_0(1 - \frac{1}{k})^t < n_0(e^{-\frac{1}{k}})^t = n \cdot e^{-\frac{t}{k}}$

  $e^x = \lim_{n\to\infty}(1 + \frac{x}{n})^n \geq 1 + x$, for $x \geq -1$, and when $x \neq 0$, the inequality holds
- With $t = k \ln n$ we have $n_t < 1$, by then we must have done!

Greedy solution of Set Cover is close to optimal

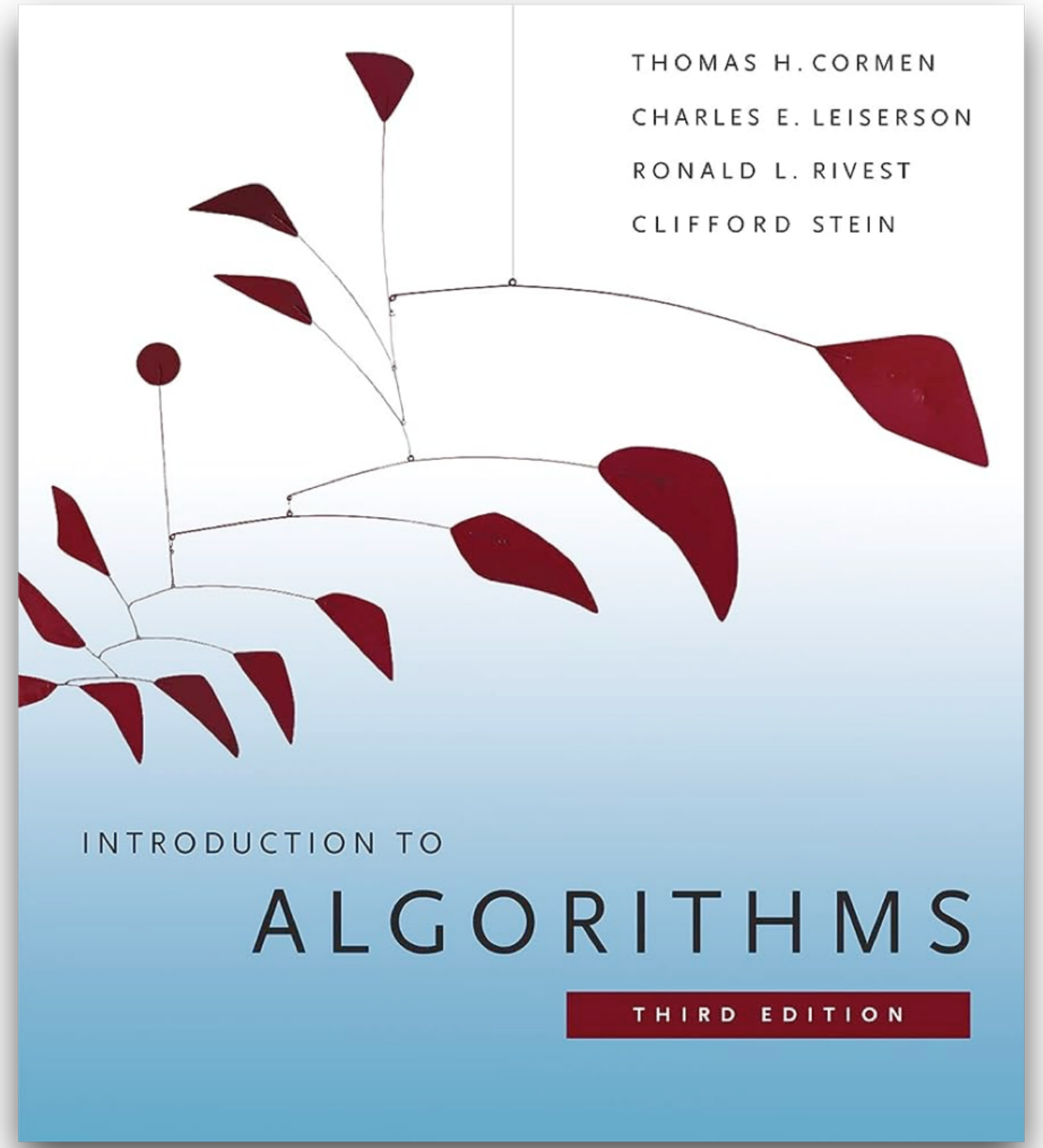
- Simple greedy strategy: Keep picking the set the covers most uncovered elements, until all elements are covered.
- Theorem Suppose the optimal solution uses k sets, then the greedy strategy will use at most $k \ln n$ sets.
- So the greedy strategy gives a $\ln n$ **approximation algorithm**, and it has efficient implementation. (Polynomial runtime.)
- Can we do better?
  - Most likely, **NO**! If we only care about efficient algorithms.
    - [Dinur & Steuer STOC14] There is no polynomial-runtime $\left(1 - o(1)\right) \cdot \ln n$ approximation algorithm unless **P** = **NP**.

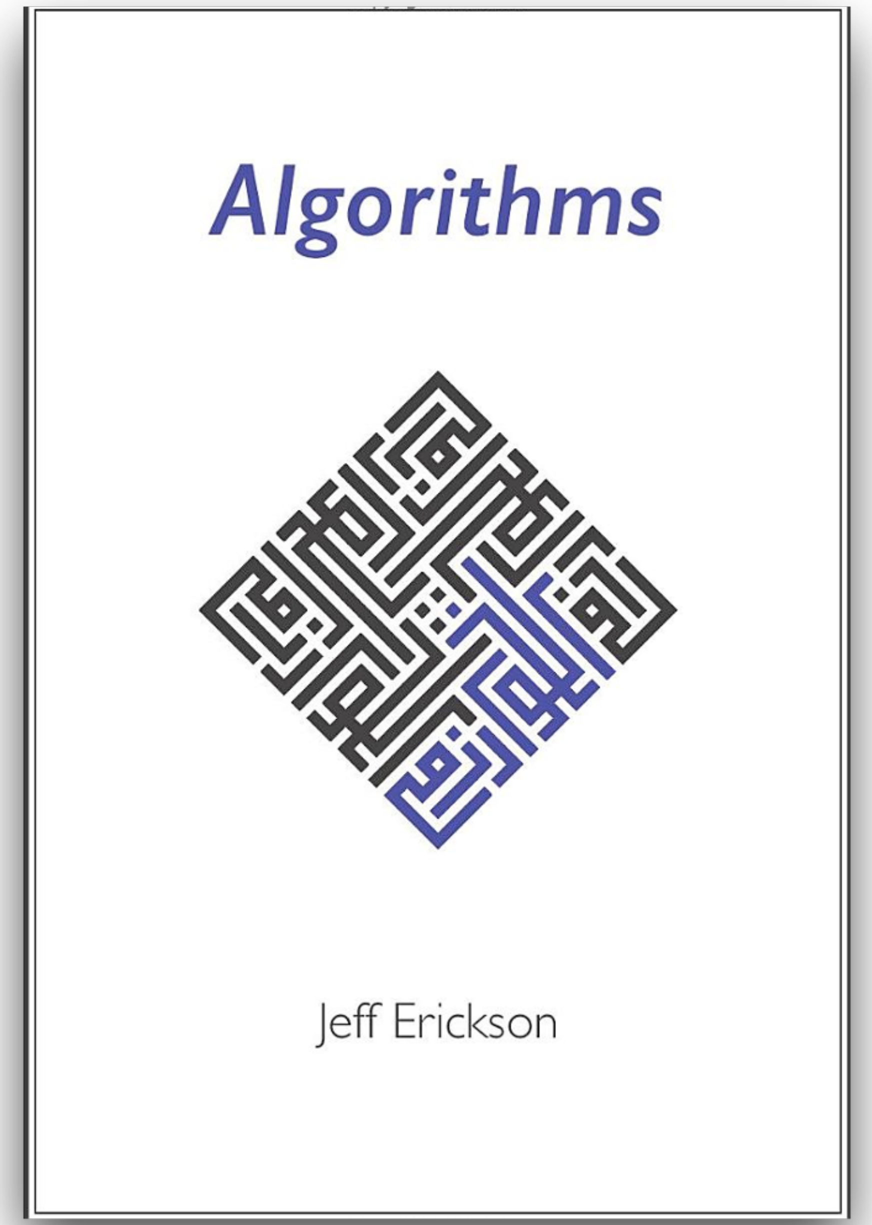
Summary

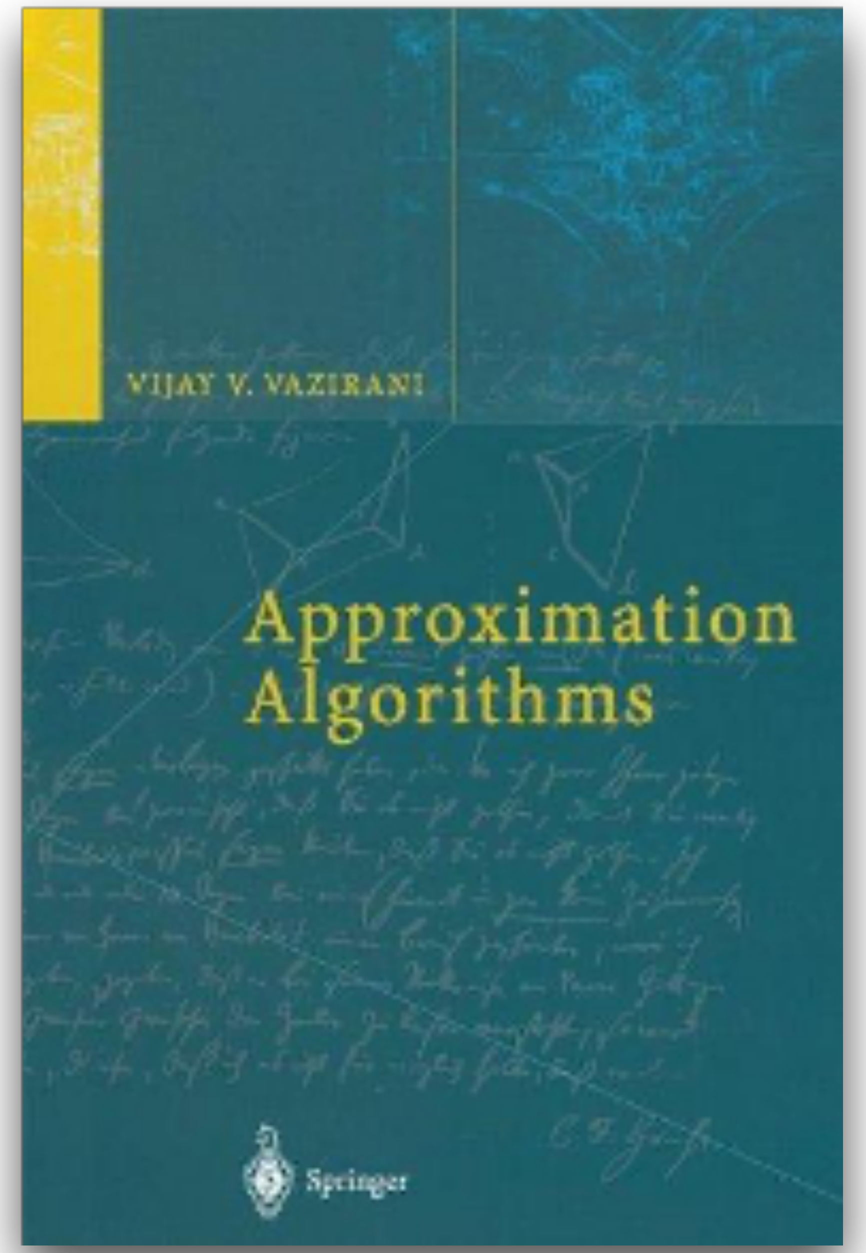
- **Basic idea of greedy strategy**: At each step when building a solution, make the choice that looks best at that moment, based on some metric.
- **Properties that make greedy strategy work**:
  - **Optimal substructure** [usually easy to prove]: optimal solution to the problem contains within it optimal solution(s) to subproblem(s).
  - **Greedy choice** [could be hard to identify and prove]: the greedy choice is contained within some optimal solution.
- Greed gives **optimal** solutions: MST, Huffman codes, …
- Greed gives **near-optimal** solutions: Set cover, …
- Greed gives **arbitrarily bad** solutions: 0-1 knapsack, …

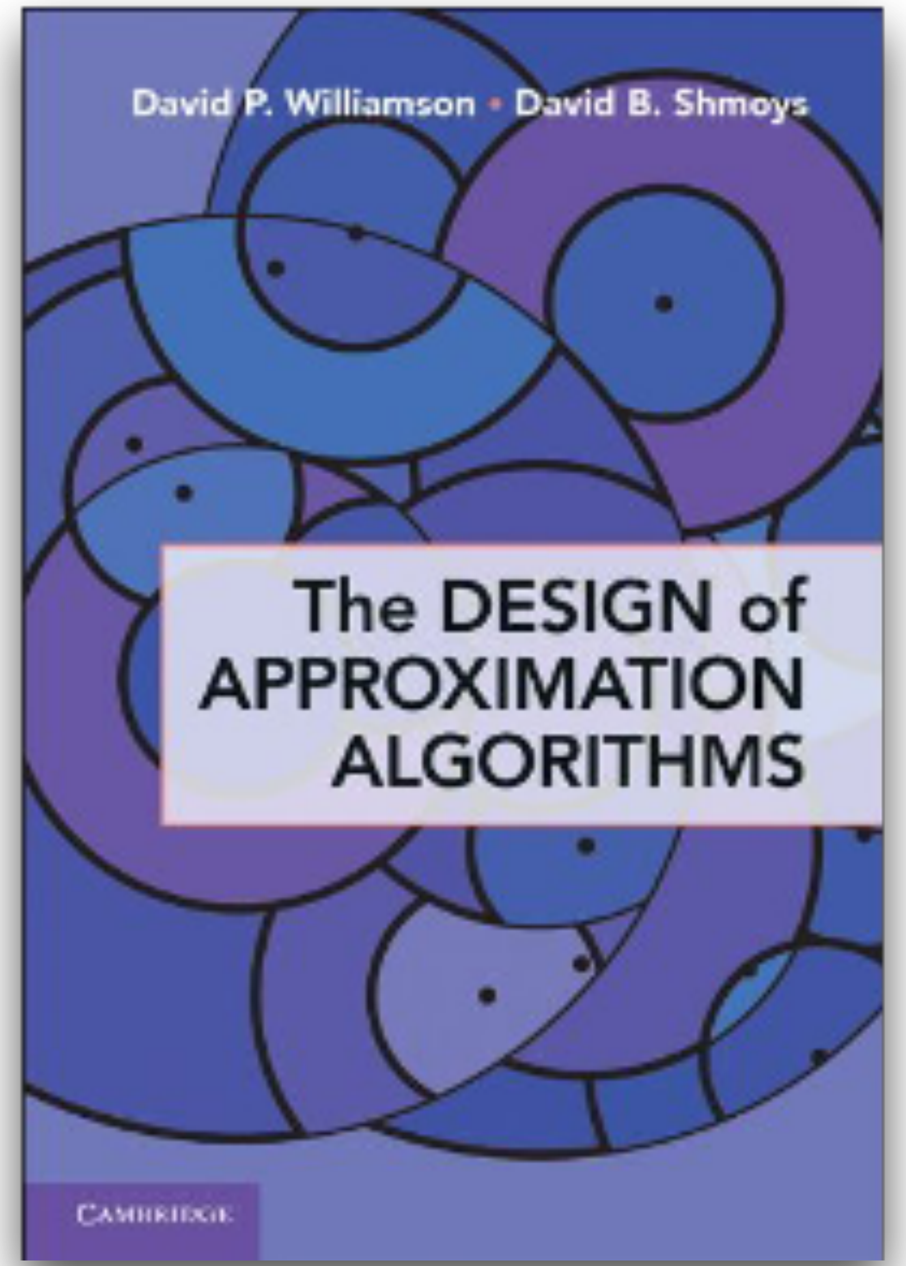
Further reading

- [CLRS] Ch.16 (16.1-16.3, 35.3)
- [Erickson v1] Ch.4 (4.5)









Refer to [Vazirani] and [Williamson & Shmoys] for more approximation algorithms