堆
Heaps

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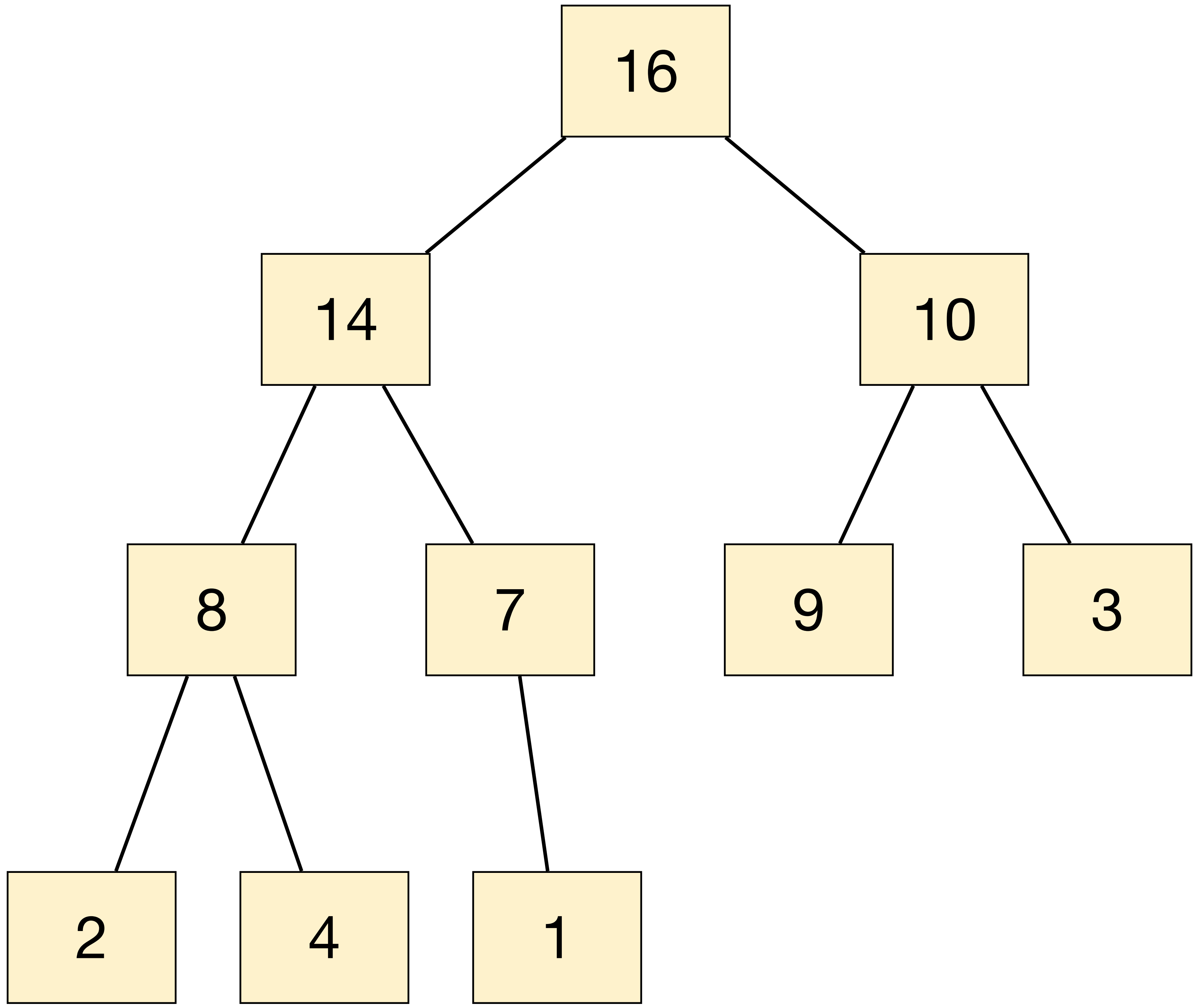
The slides are mainly adapted from the original ones shared by Chaodong Zheng and Kevin Wayne.Thanks for their supports!

Heap

- In computer science, a ***heap*** is data structure which means "a disorganized pile."
  - In fact, this word has other meanings in computer science, which refers to *heap memory* used for dynamic memory allocation. This topic, however, is **unrelated** to the data structure in this course!

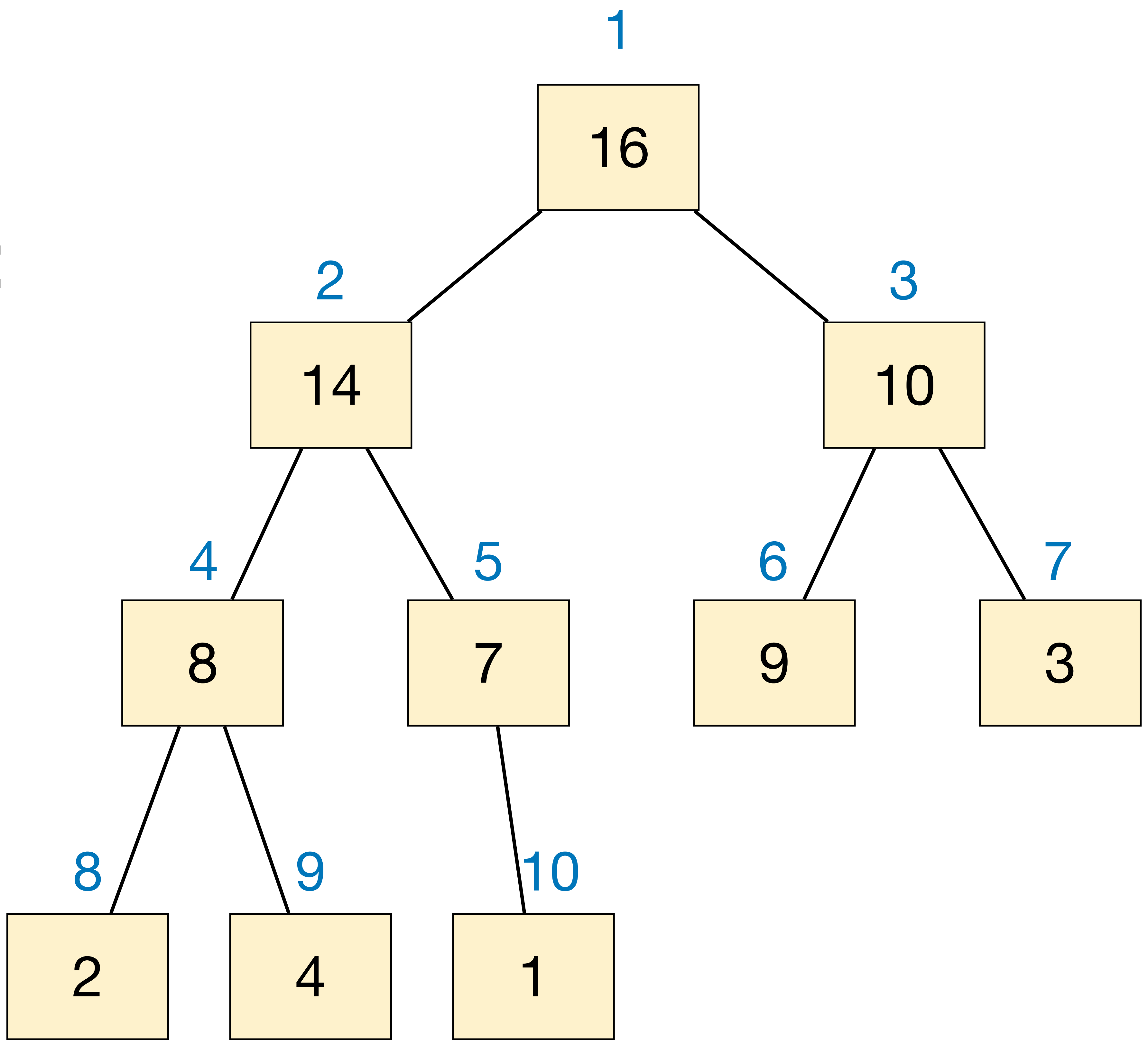
Binary Heap

- A binary heap is a **complete binary tree**, in which each node represents an item.
  - A complete binary tree is a binary tree in which every level, except possibly the last, is completely filled, and all nodes in the last level are as far left as possible.
- Values in the nodes satisfy **heap-property**.
  - Max-heap: for each node except root, value of that node $\leq$ value of its parent.
  - Min-heap: for each node except root, value of that node $\geq$ value of its parent.



Binary Heap

- We can use an array to represent a binary heap. Obtaining parent and children are easy:
  - Parent of node u : $\lfloor idx_u/2 \rfloor$
  - Left child of u : $2 \cdot idx_u$
  - Right child of u : $2 \cdot idx_u + 1$
  - All in $O(1)$ time!



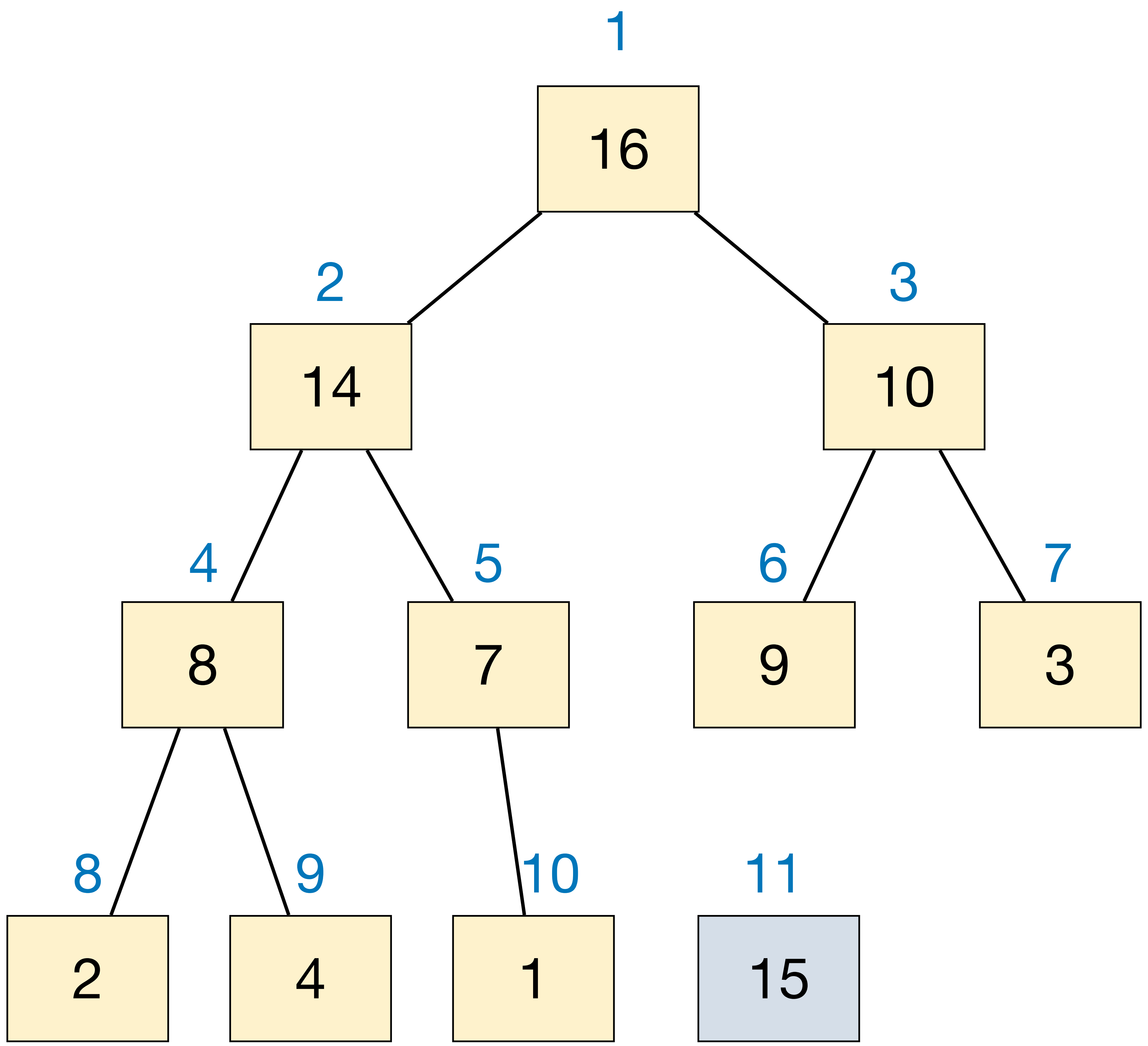
1	2	3	4	5	6	7	8	9	10
16	14	10	8	7	9	3	2	4	1

Common operations of Binary Max-Heap

- Consider max-heap as an example. (Min-heap is similar.)
- Most common operations:
  - `HeapInsert`: insert an element into the heap.
  - `HeapGetMax`: return the item with maximum value. Runtime is $O(1)$
  - `HeapExtractMax`: remove the item with maximum value from the heap and return it.
- Other operations (which we'll see later)…

Max-Heap — HeapInsert

- Insert an item into a binary max-heap represented by an array.
  - Simply put the item to the end of the array.



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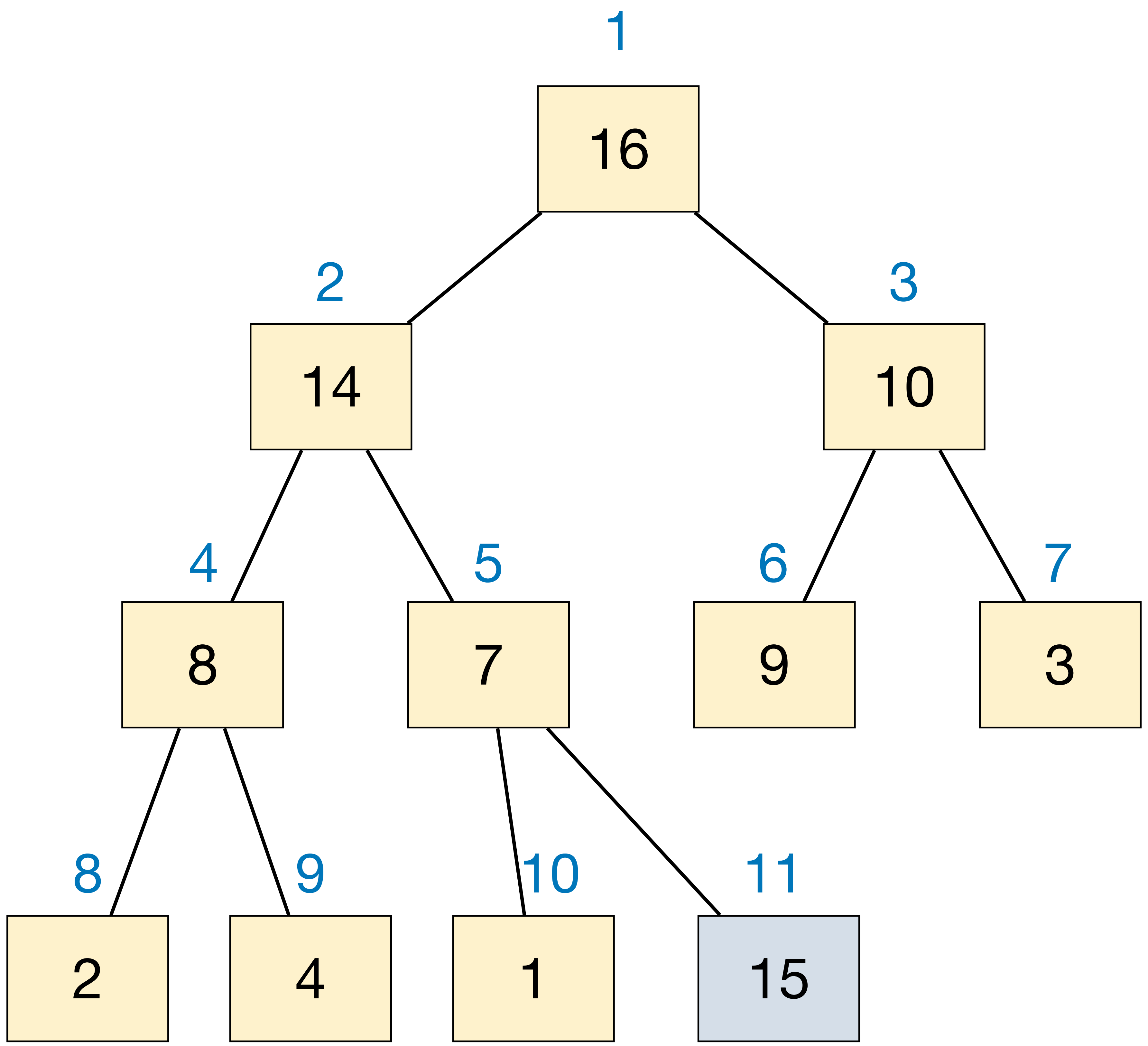

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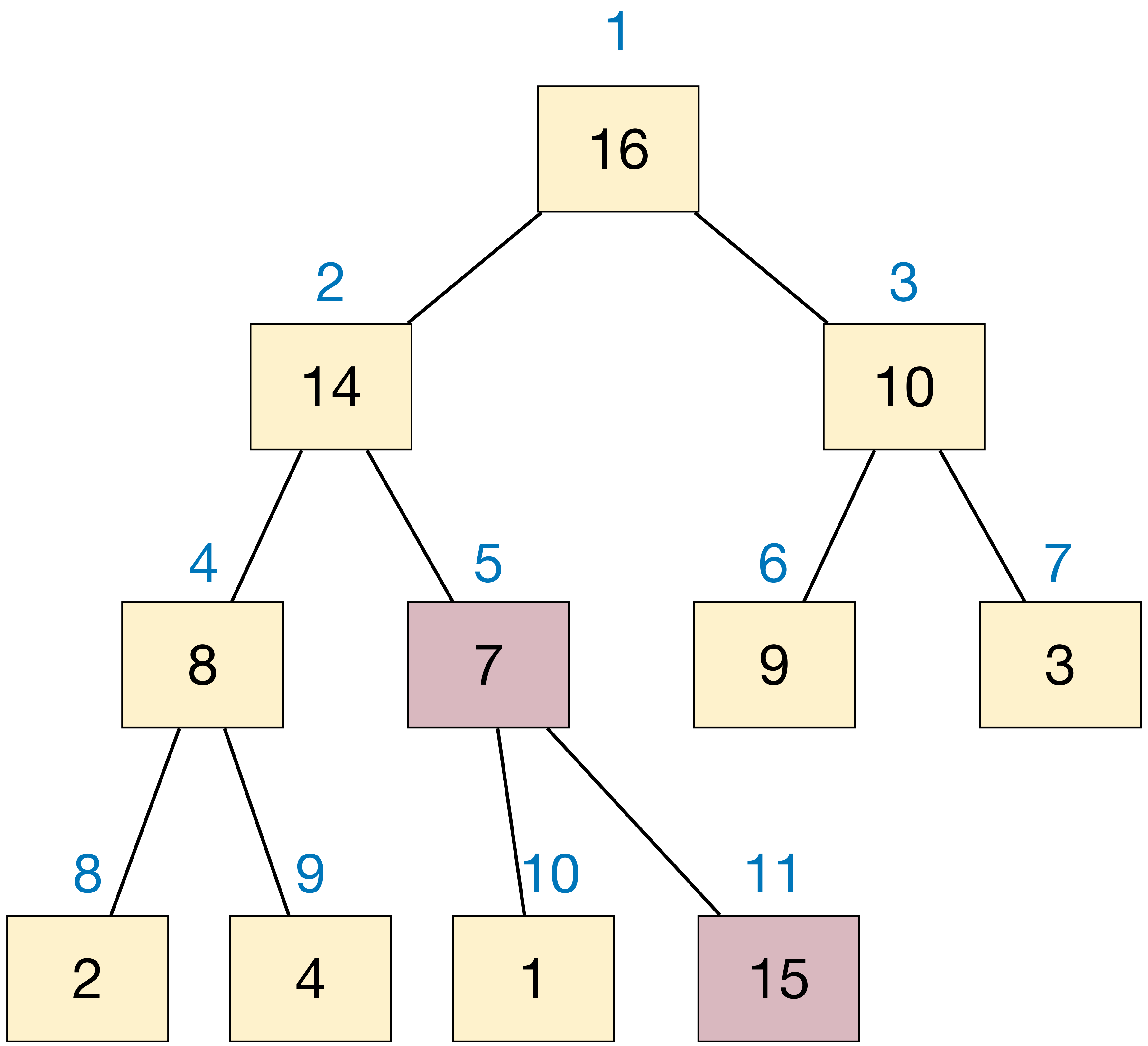

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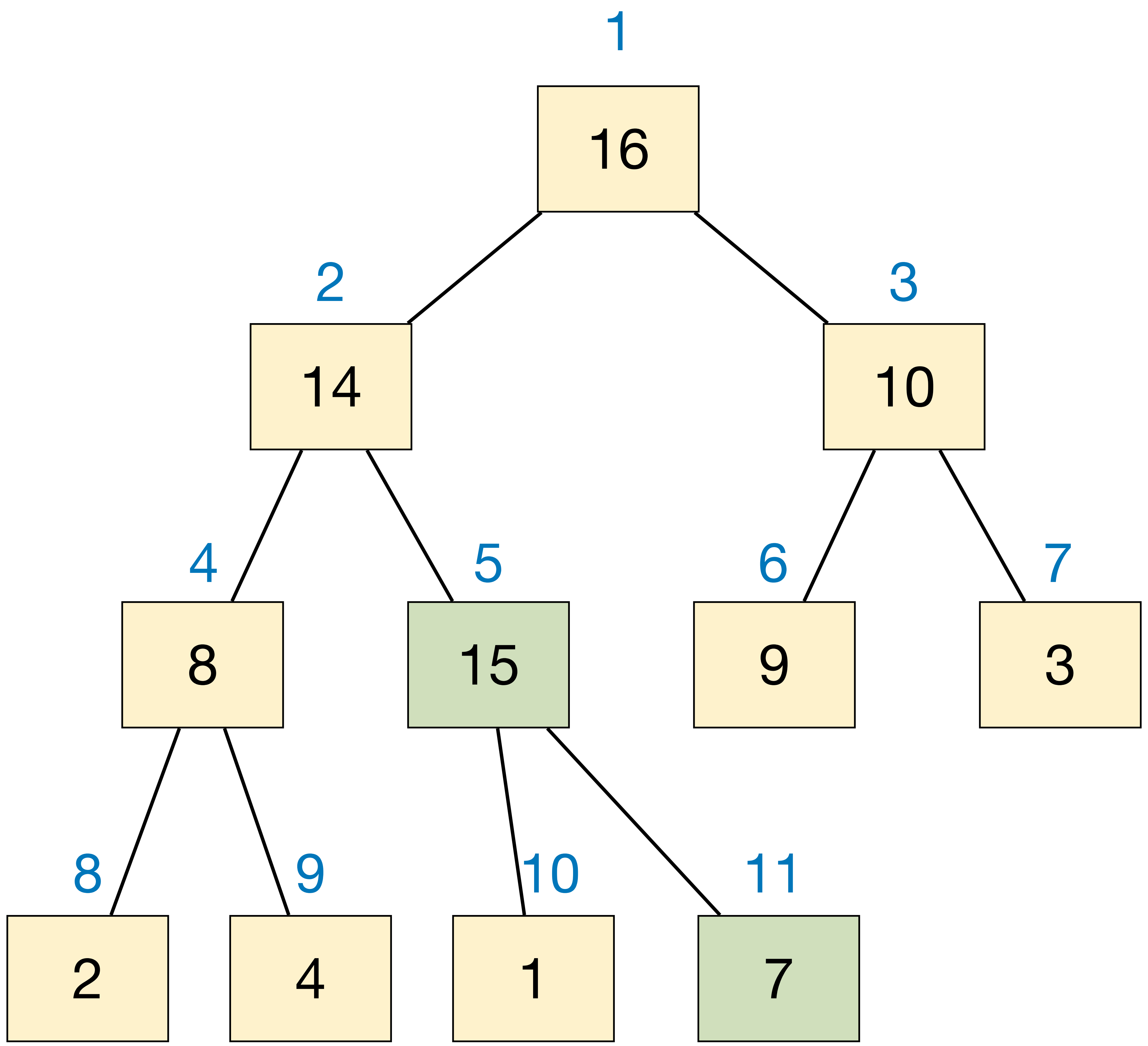

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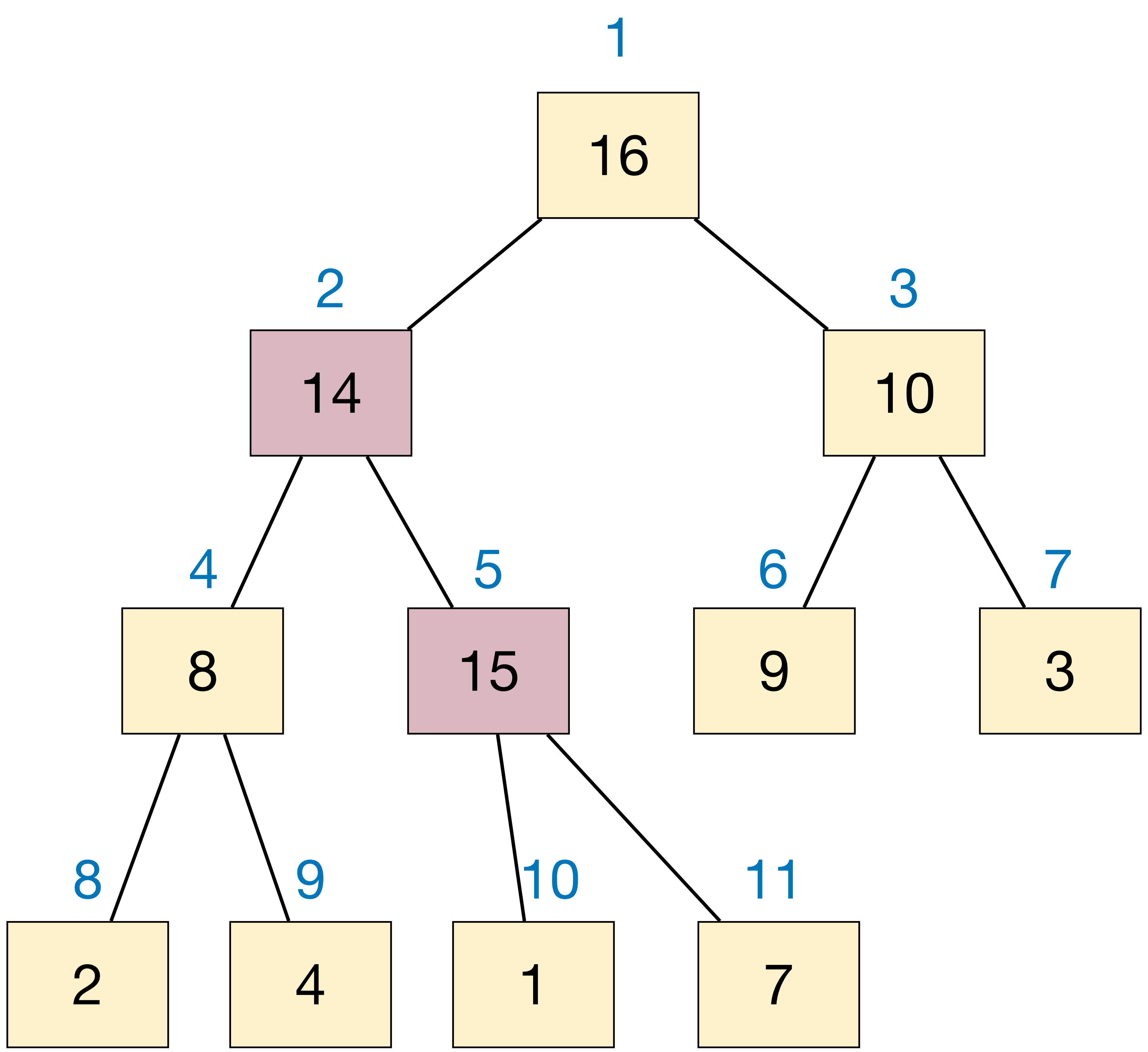

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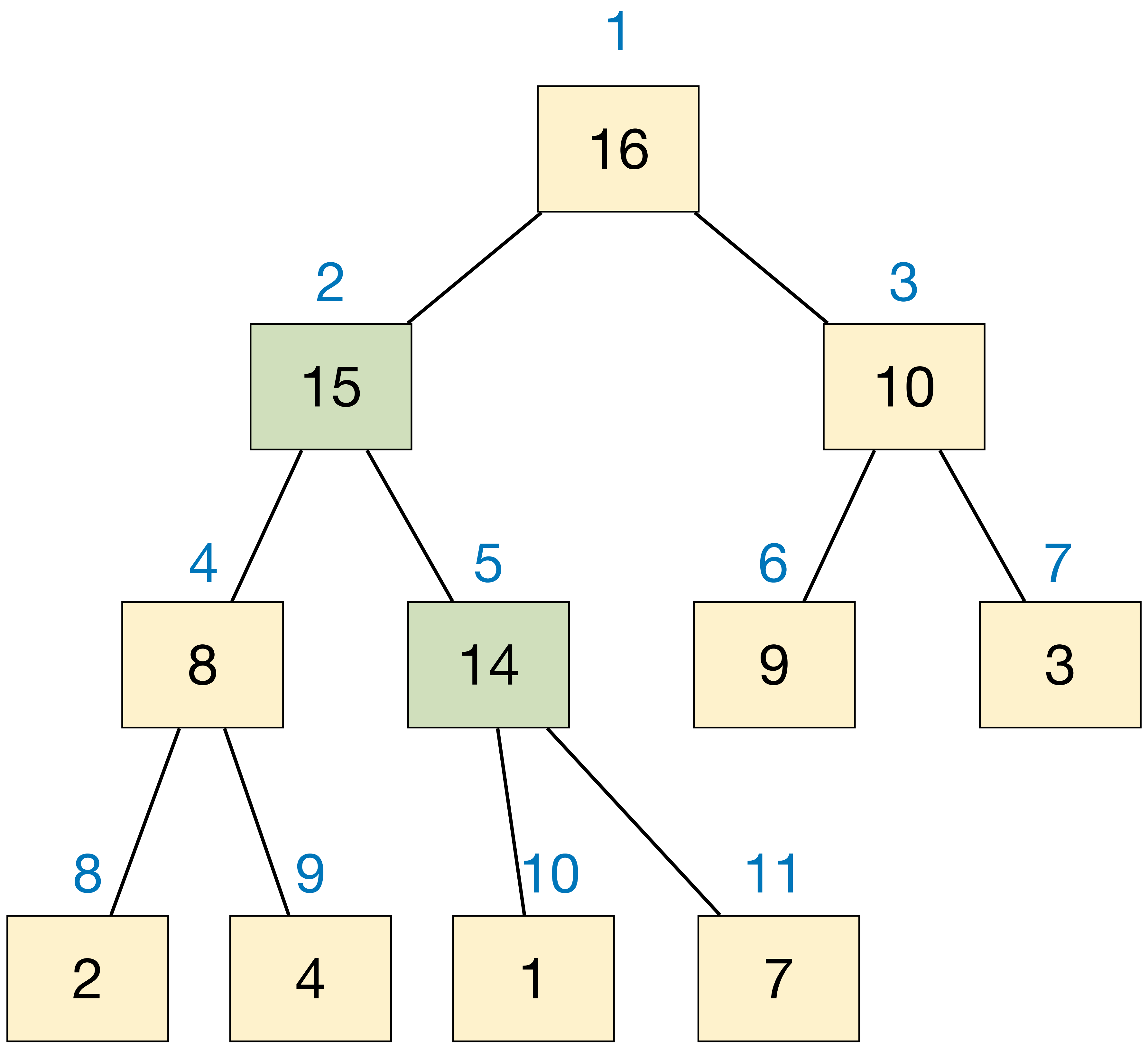

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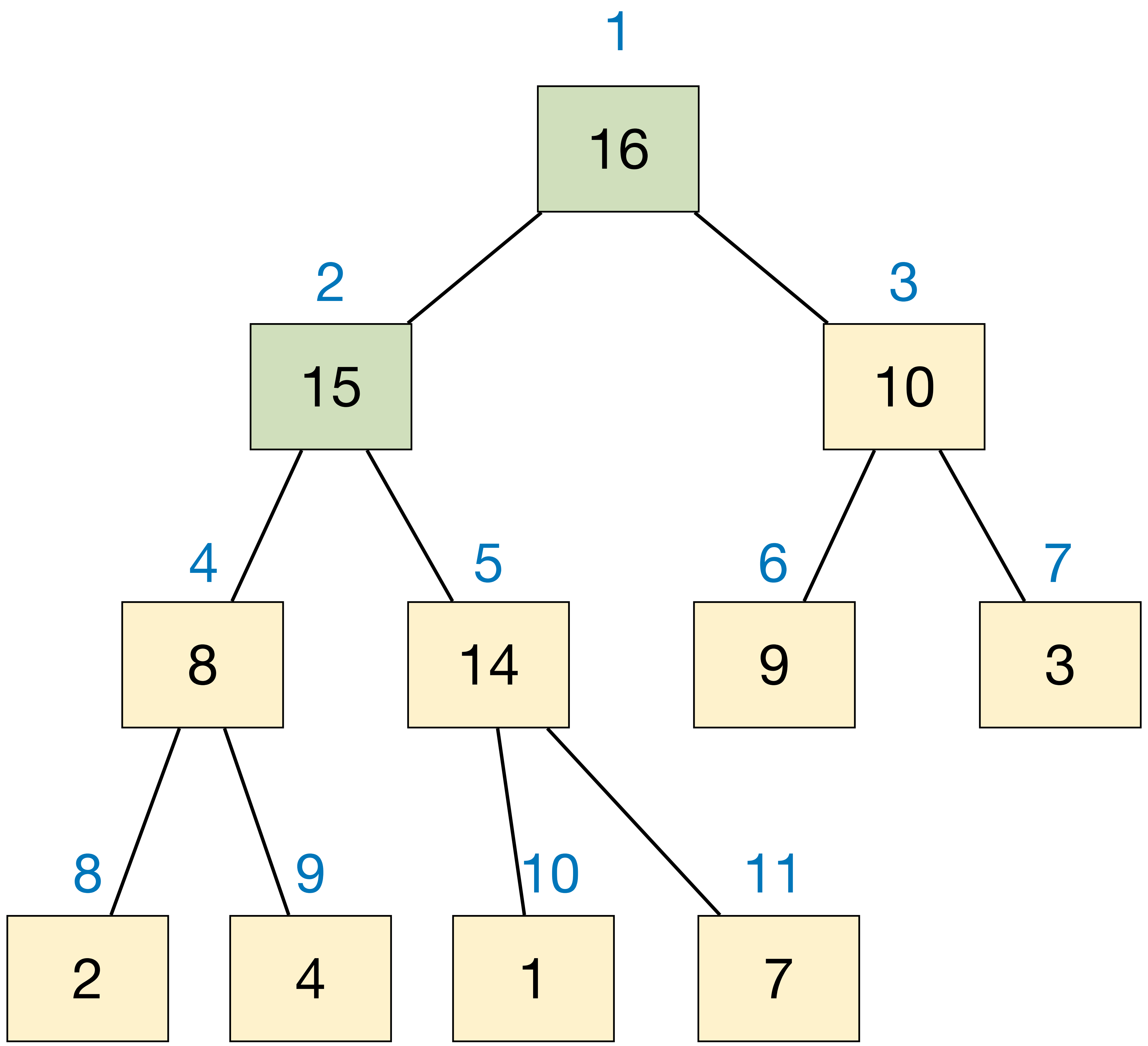

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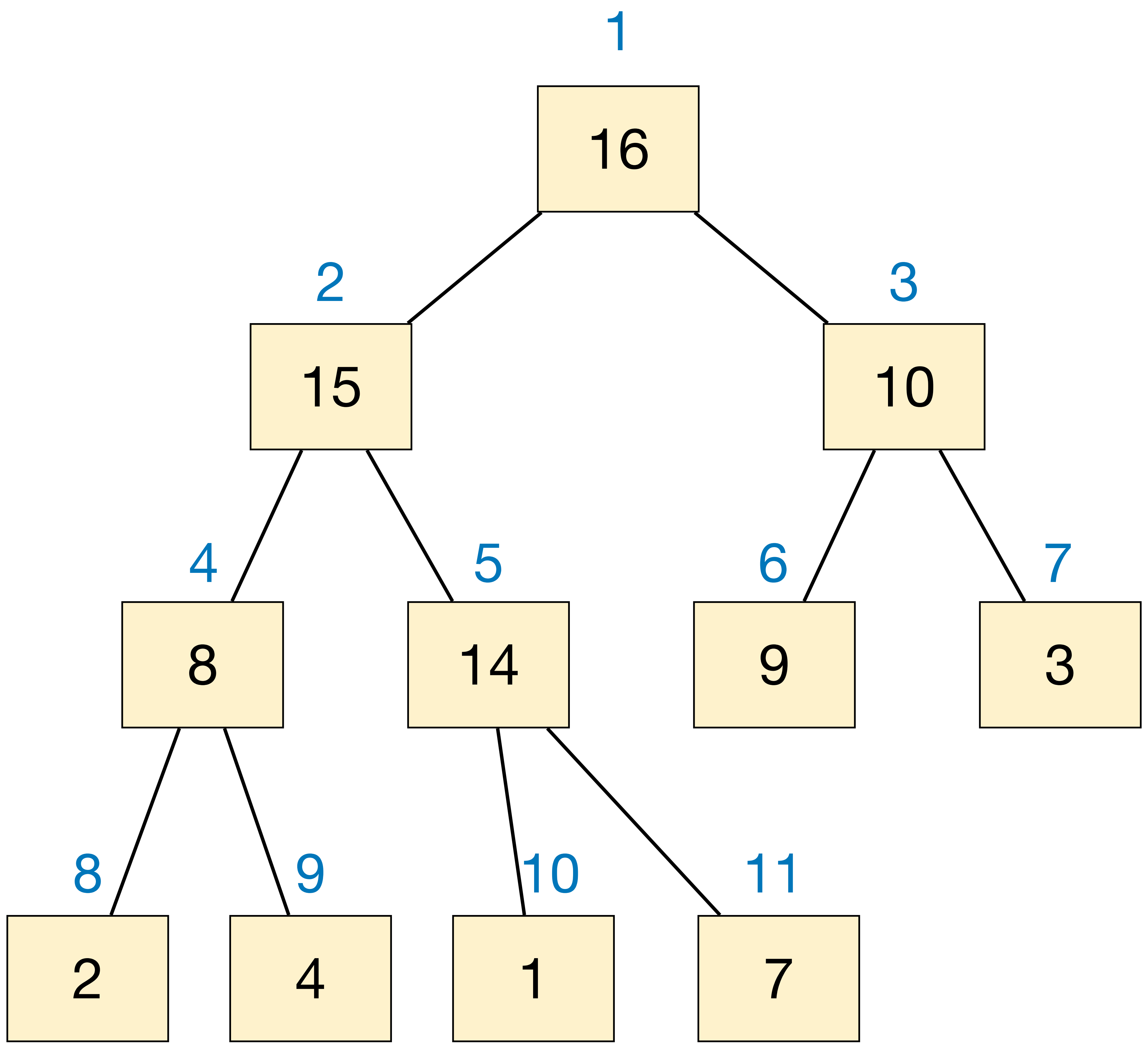
Max-Heap — HeapInsert

HeapInsert(A, x):

```
heap_size += 1
A[heap_size] := x
idx := heap_size
while idx > 1 and  A[Floor (idx / 2)] <  A[idx]
    Swap (A[Floor (idx / 2)],  A[idx])
    idx := Floor (idx / 2)
```

Runtime is $O(\lg n)$



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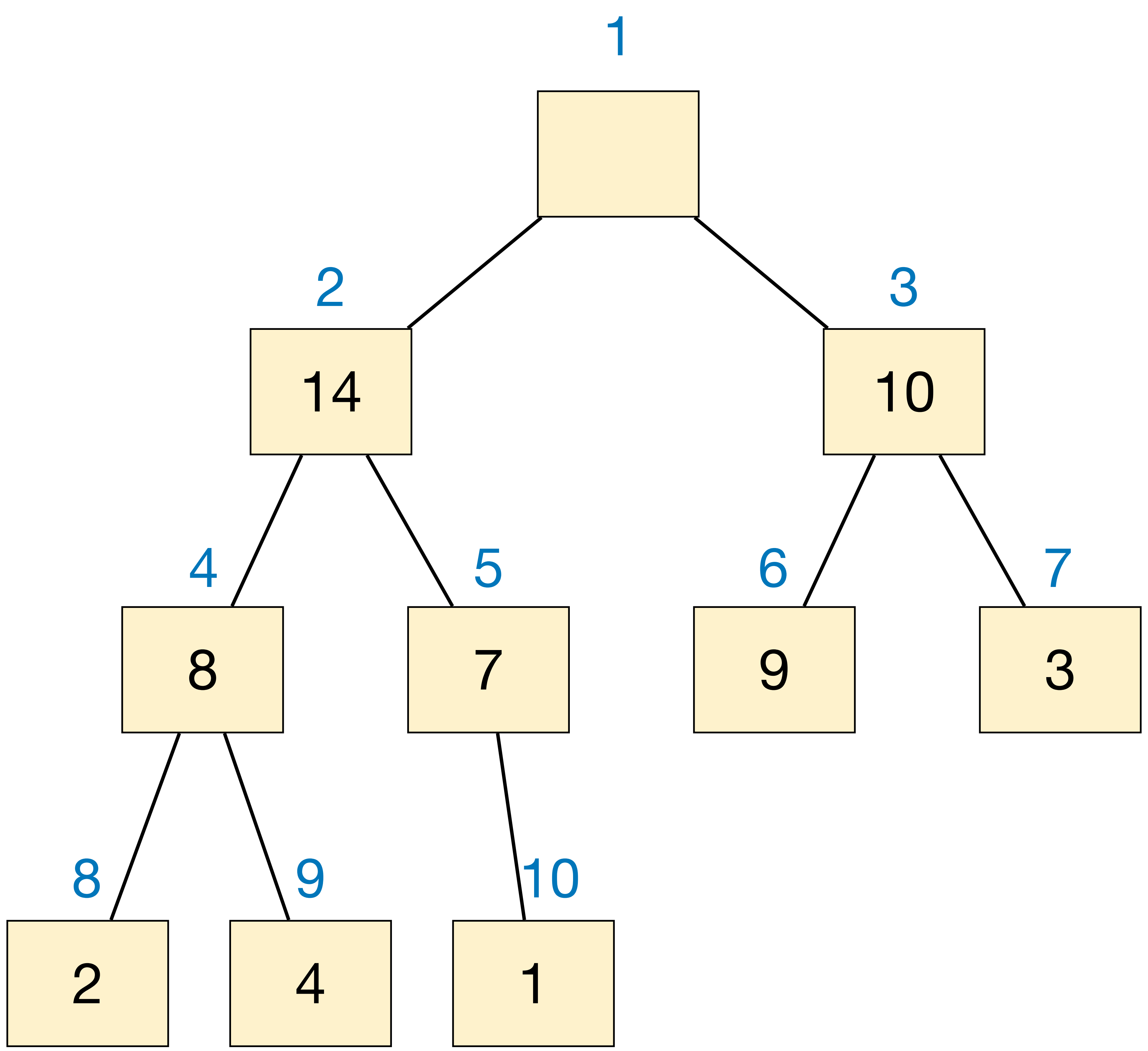

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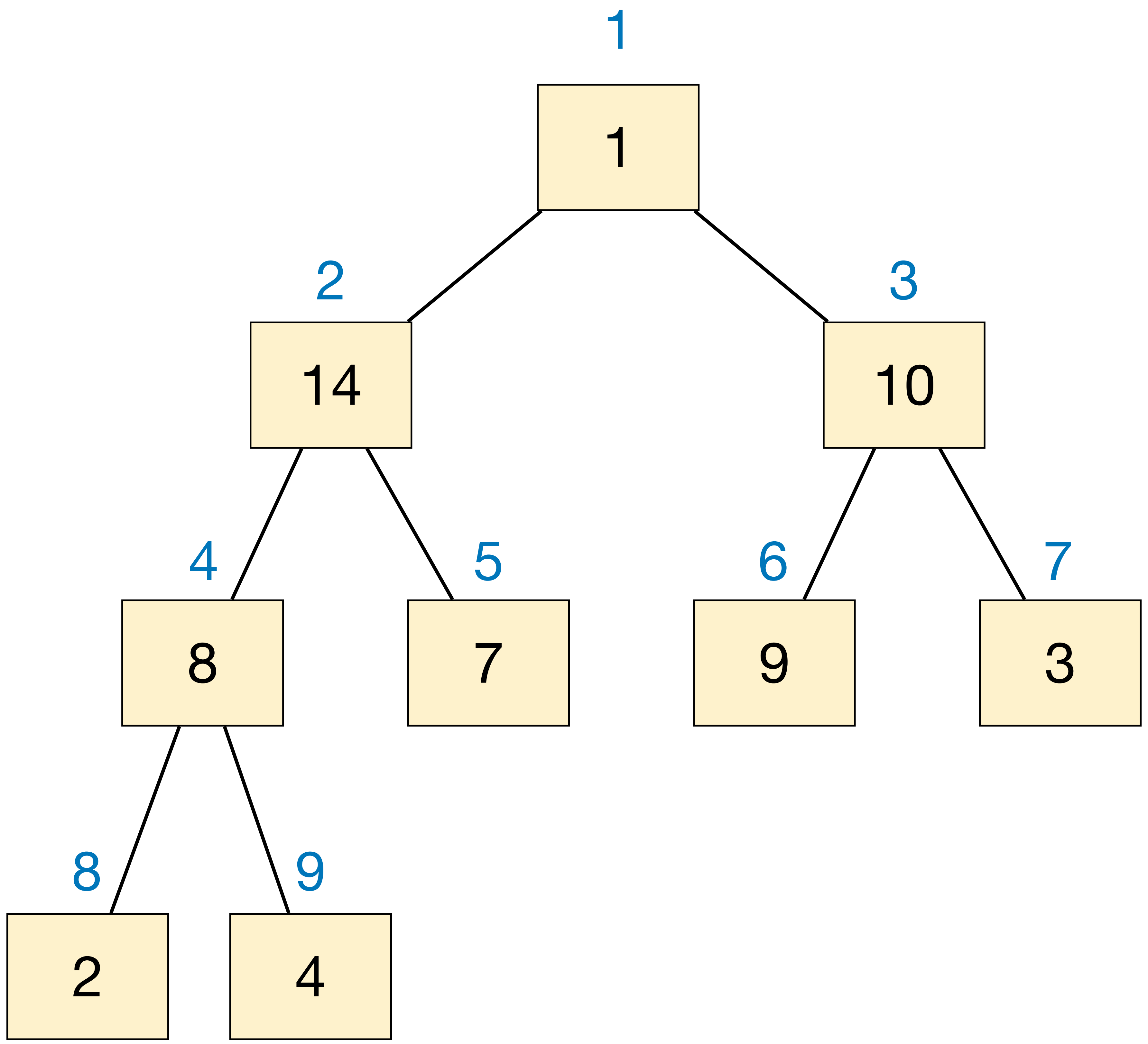

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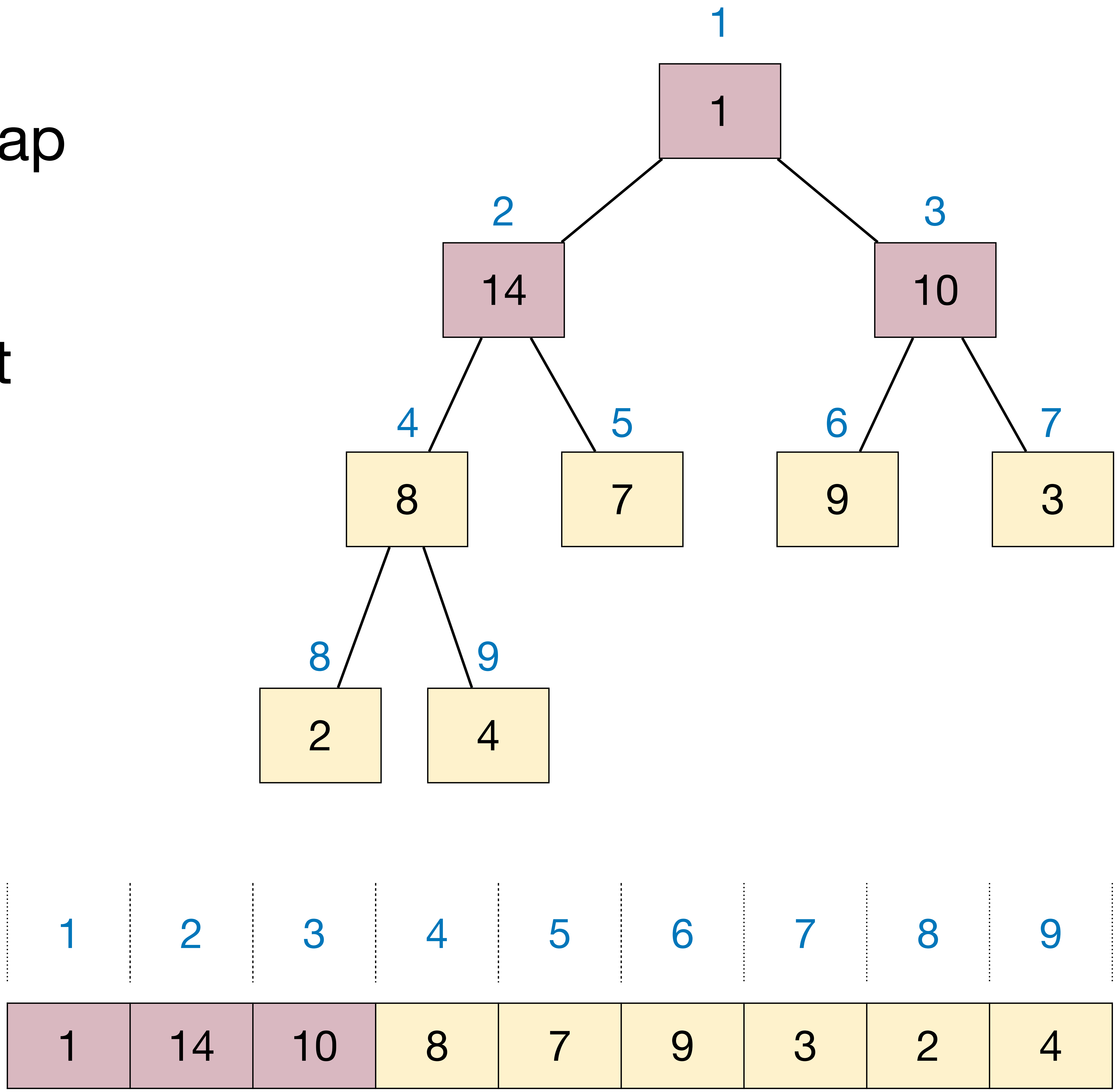

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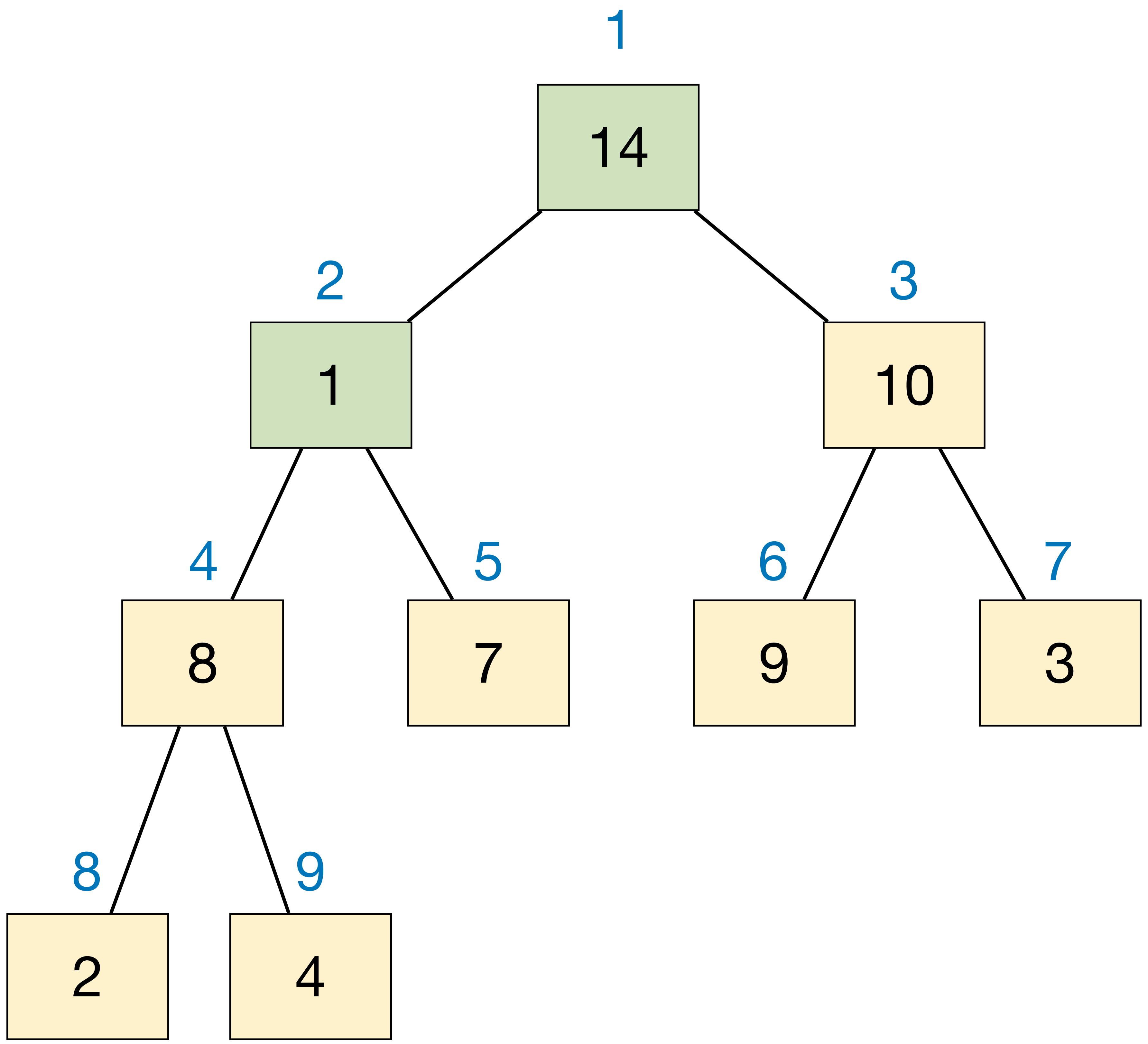

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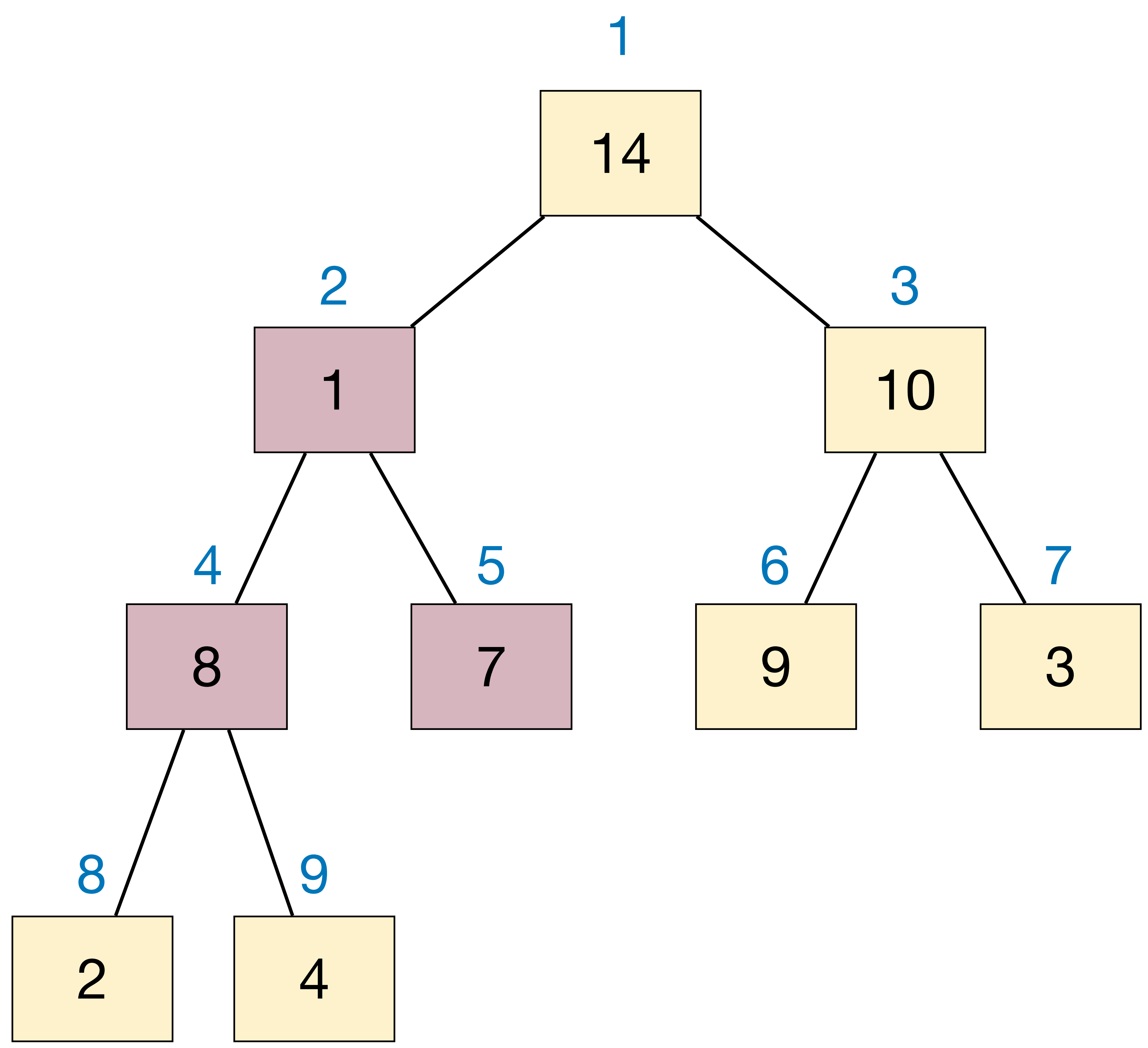

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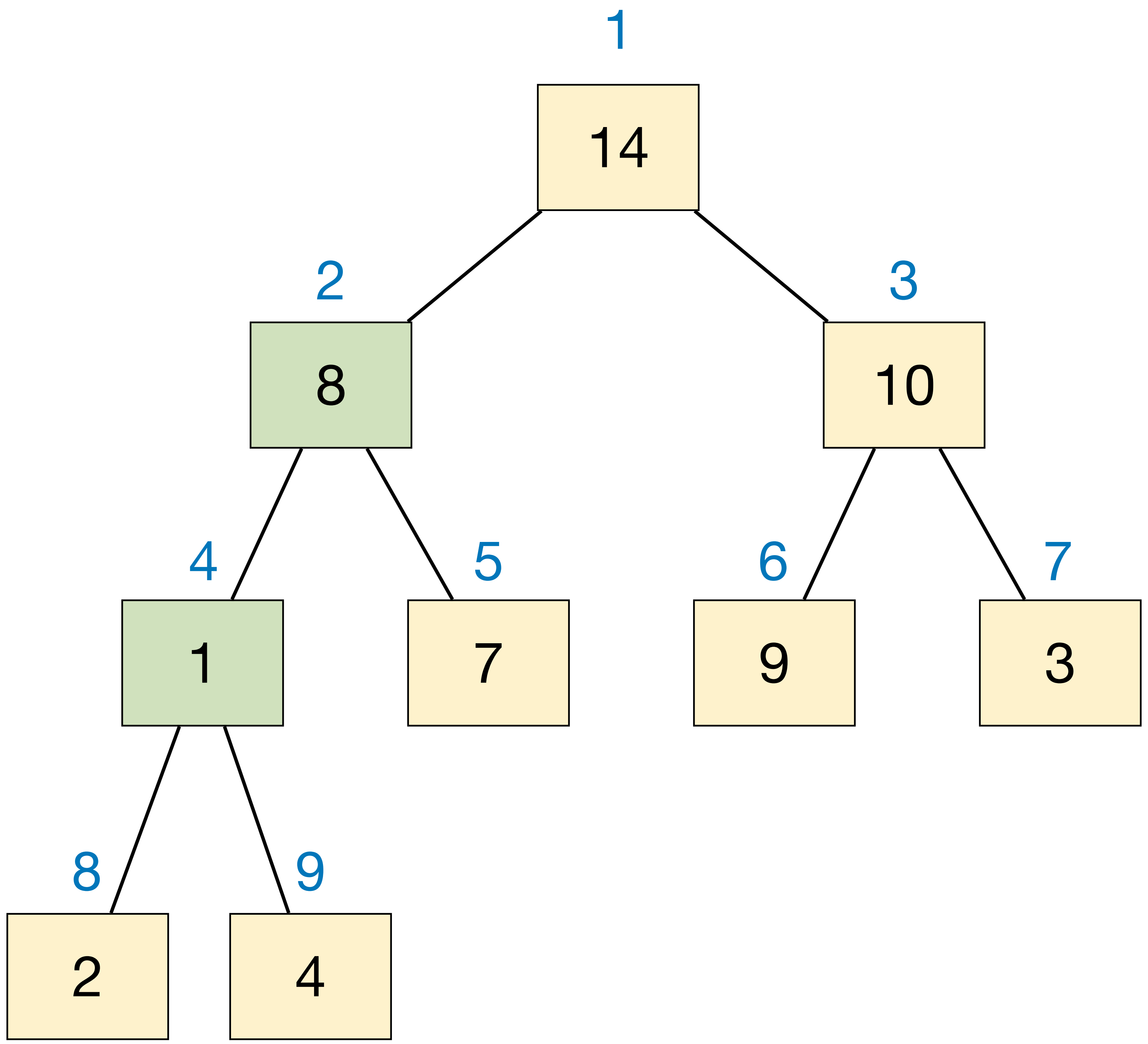

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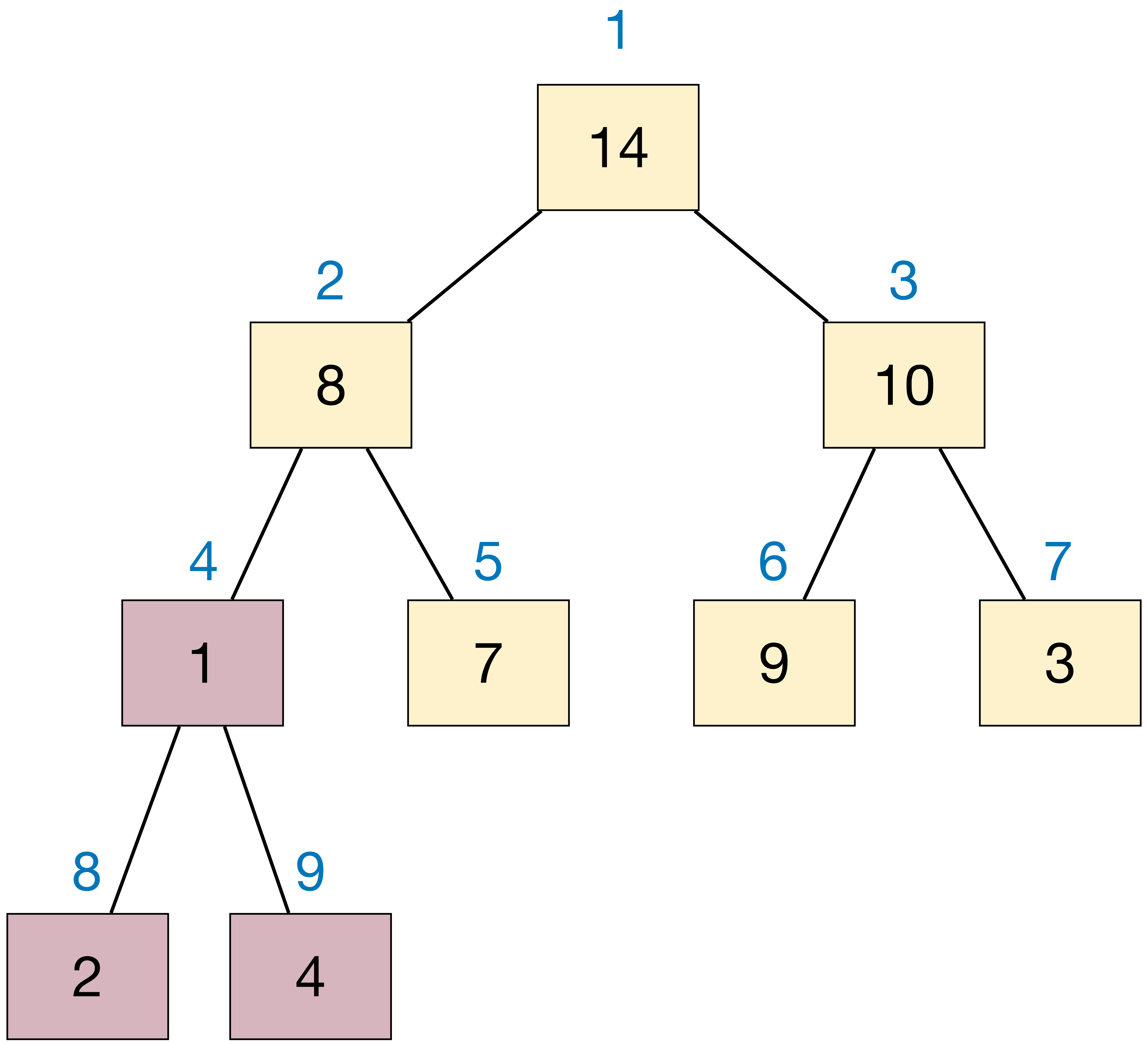

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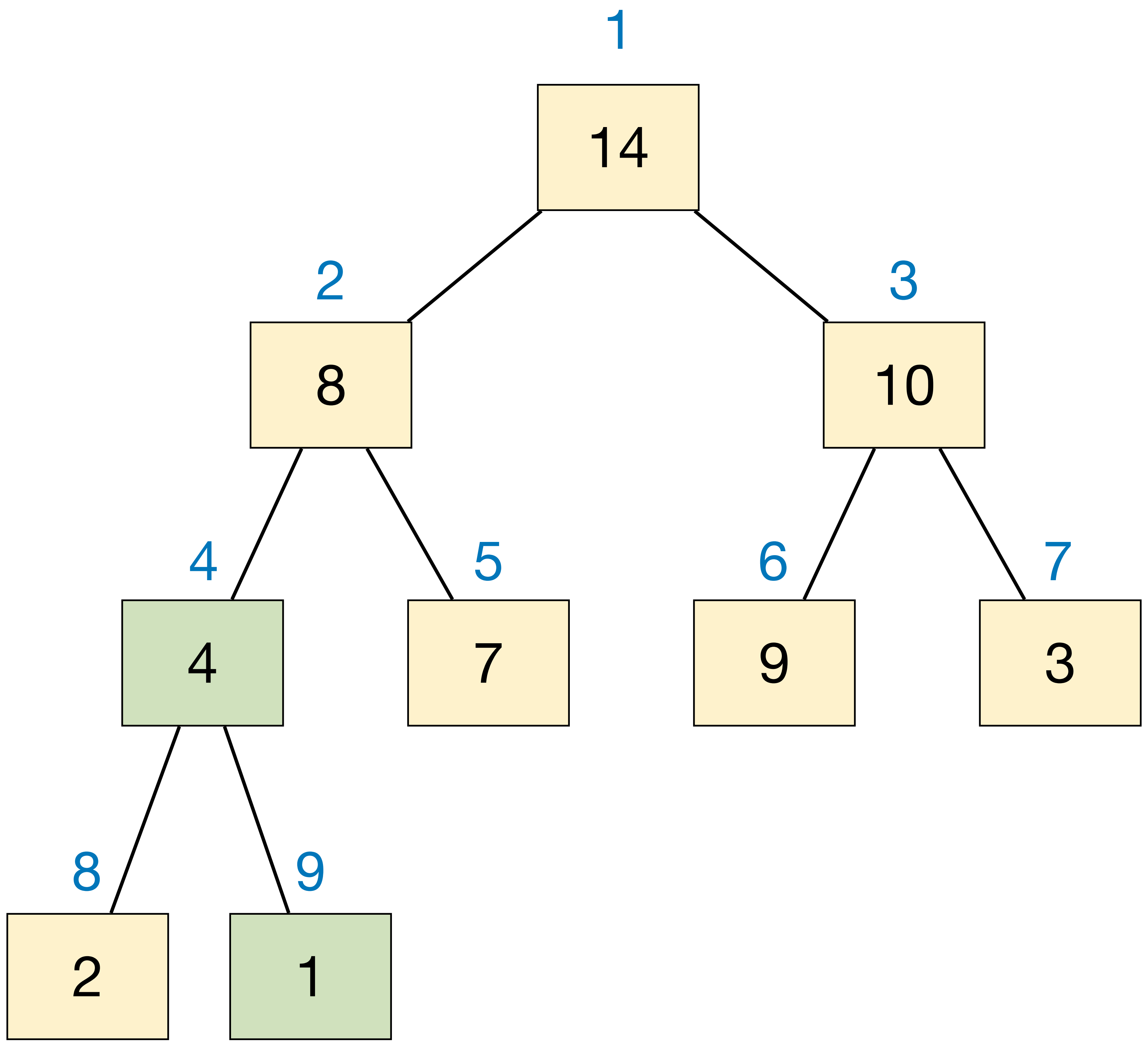

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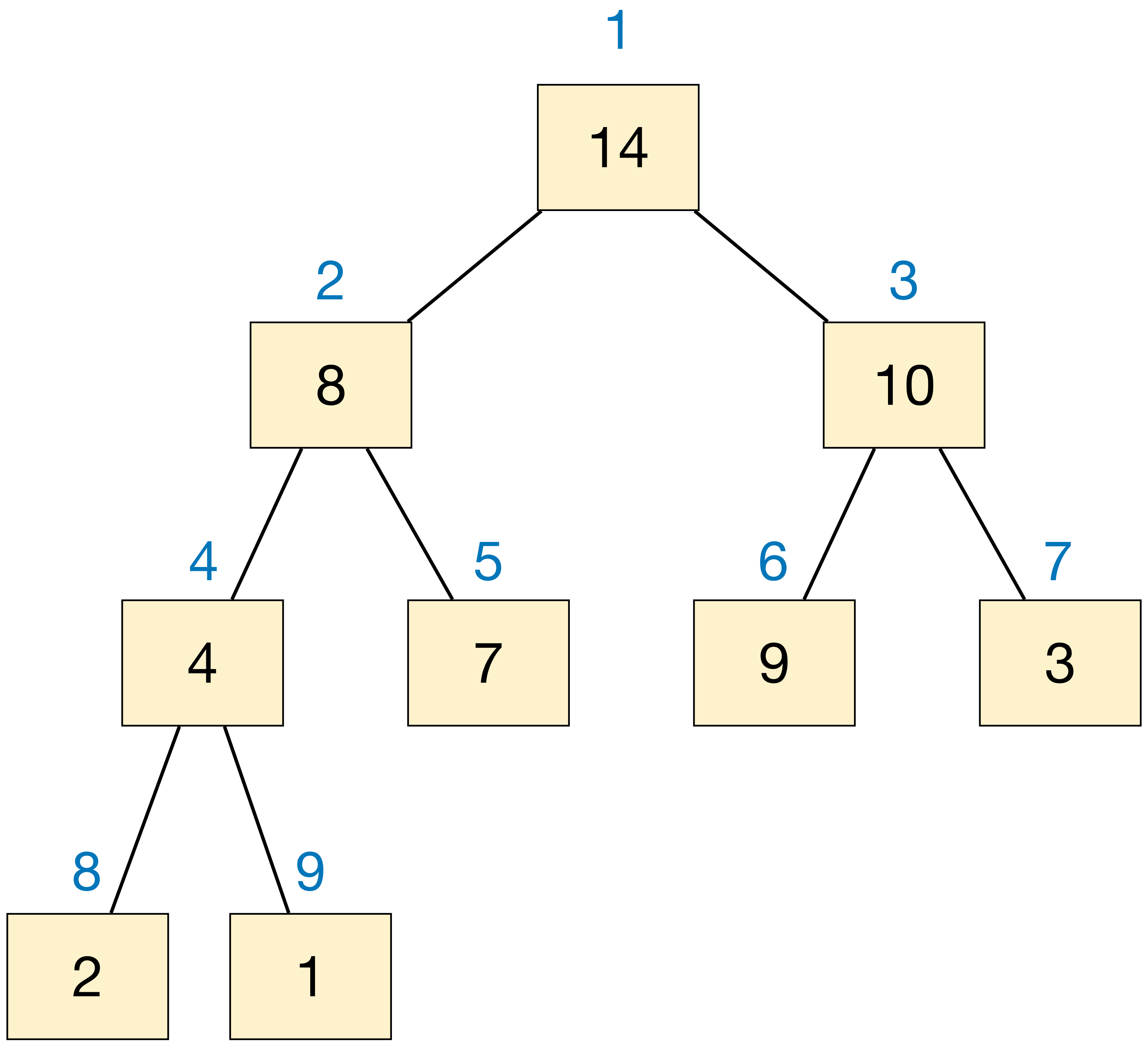

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Max-Heap — HeapExtractMax

HeapExtractMax(A):

$max\_item := A[1]$

$A[1] = A[heap\_size\text{--}]$

$MaxHeapify(1, A)$

return max_item

MaxHeapify(idx, A):

$idx\_l :=\ 2*idx,\ \ idx\_r := 2*idx + 1$

$idx\_max :=\ (\ idx\_l <= heap\_size$ **and** $A[idx\_l] > A[idx]\ )\ ?\ idx\_l : idx$

$idx\_max :=\ (\ idx\_r <= heap\_size$ **and** $A[idx\_r] > A[idx\_max]\ )\ ?\ idx\_r : idx\_max$

if $idx\_max \mathrel{!}= idx$

    $Swap\ (A[idx\_max], A[idx])$

    $MaxHeapify(idx\_max, A)$

Runtime is $O(\lg n)$

Application of heaps:

Priority Queue

Priority Queue

- Recall the `Queue` ADT represents a collection of items to which we can add items and remove the next item.
  - `Add(item)`: add item to the queue.
  - `Remove()`: remove the next item `y` from queue, return `y`.
- The queuing discipline decides which item to be removed.
  - First-in-first-out queue (FIFO Queue)
  - Last-in-first-out queue (LIFO Queue, Stack)
  - **Priority queue**: each item associated with a **priority**, **`Remove`** always deletes the item with max (or min) priority.

Priority Queue

- Use binary heap to implement priority queue
  - `Add(item):HeapInsert(item)`
  - `Remove():HeapExtractMax()`
  - Other operations: `GetMax()`, `UpdatePriority(item, val)`
  - All these operations finish within $O(\lg n)$ time
- Application of priority queues
  - Scheduling, Event simulation, …
  - Used in more sophisticated algorithms (will see them later…)

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HeapSort

Take an array and make it a max-heap.

HeapSort(I):

$heap := BuildMaxHeap(I)$
for $i := n$ **down to** 2
  $cur\_max := heap.HeapExtractMax()$
  $I[i] := cur\_max$

1. Keep a copy of the root item
2. Remove last item and put it to root
3. Maintain heap property
4. Return the copy of the root item

In each iteration:

Place **one** item in the array to its final position.

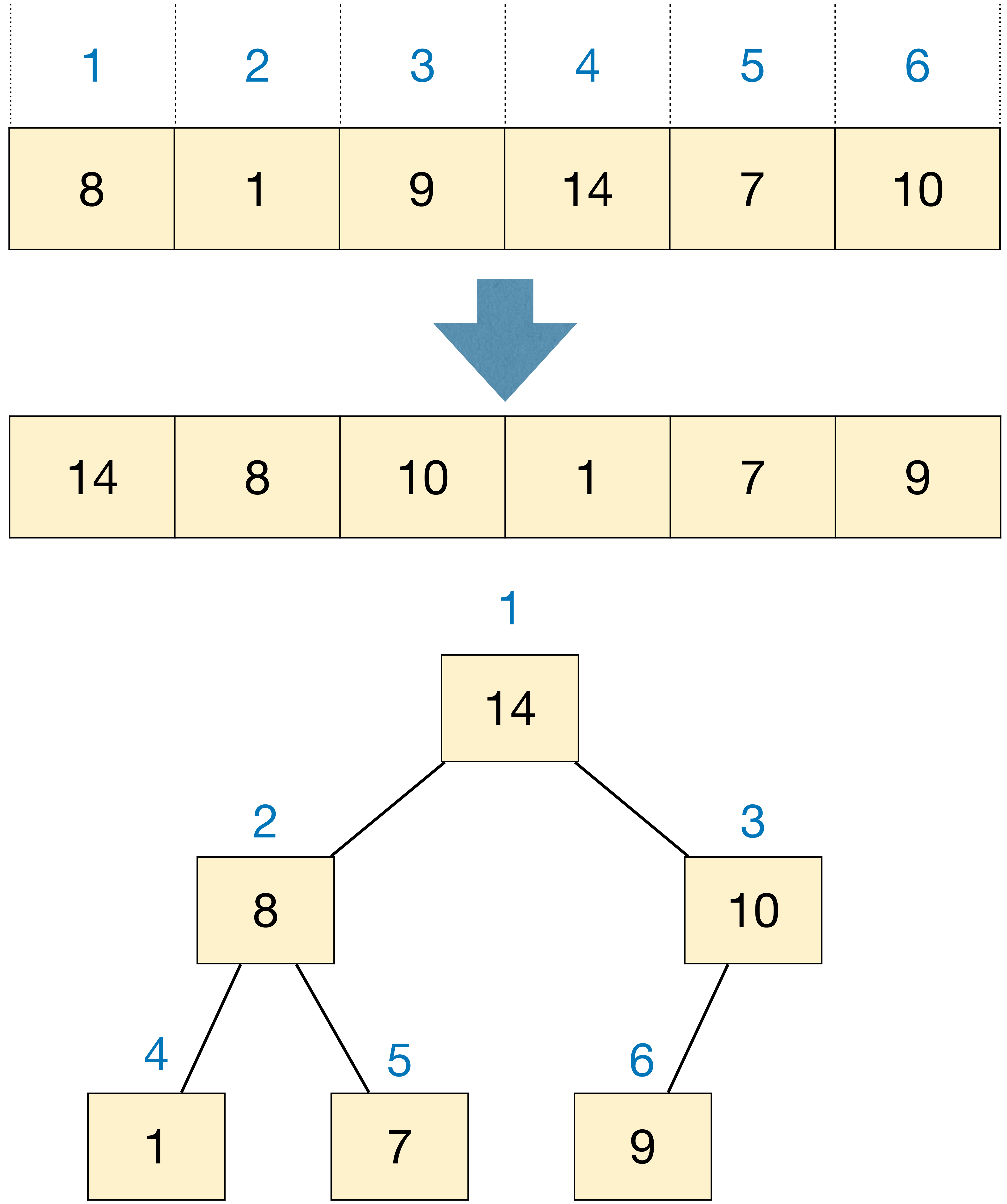
This **one** is the max item in current heap.

That is, place i^{th} biggest item to position $n - i + 1$.

The loop invariant:

The largest i elements are already in their correct positions.

HeapSort



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for $i := n$ **down to** 2

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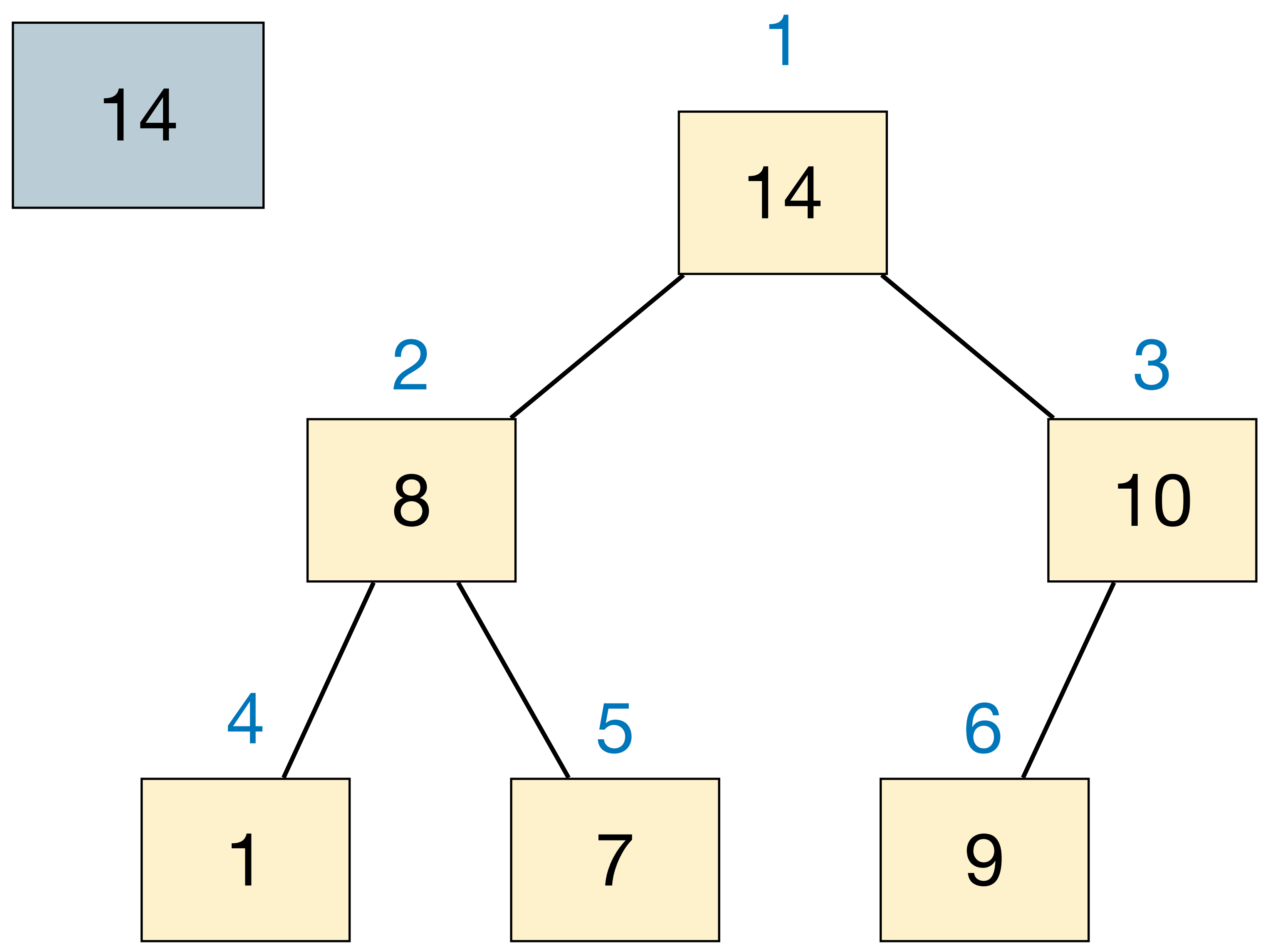
$i = 6$



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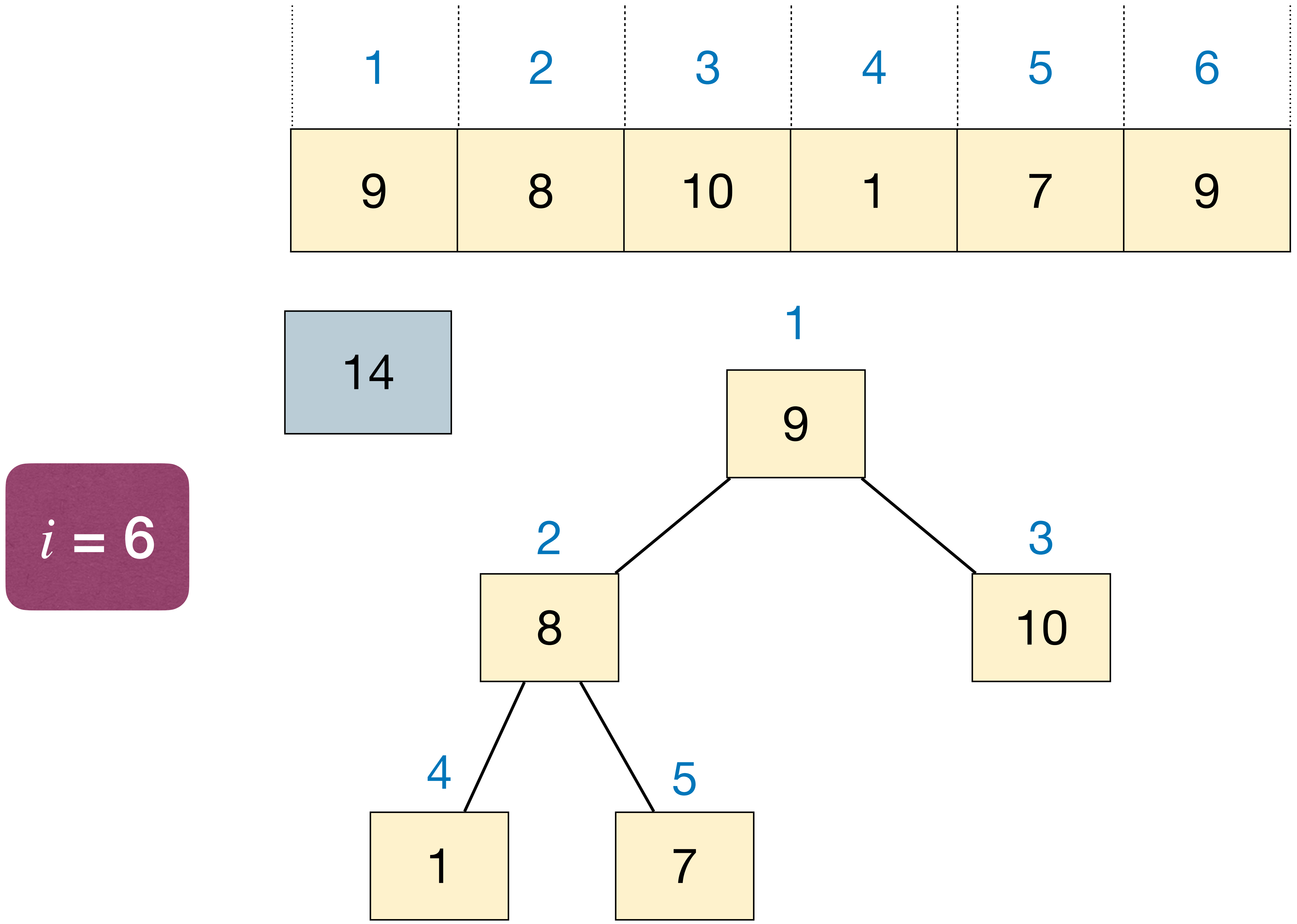

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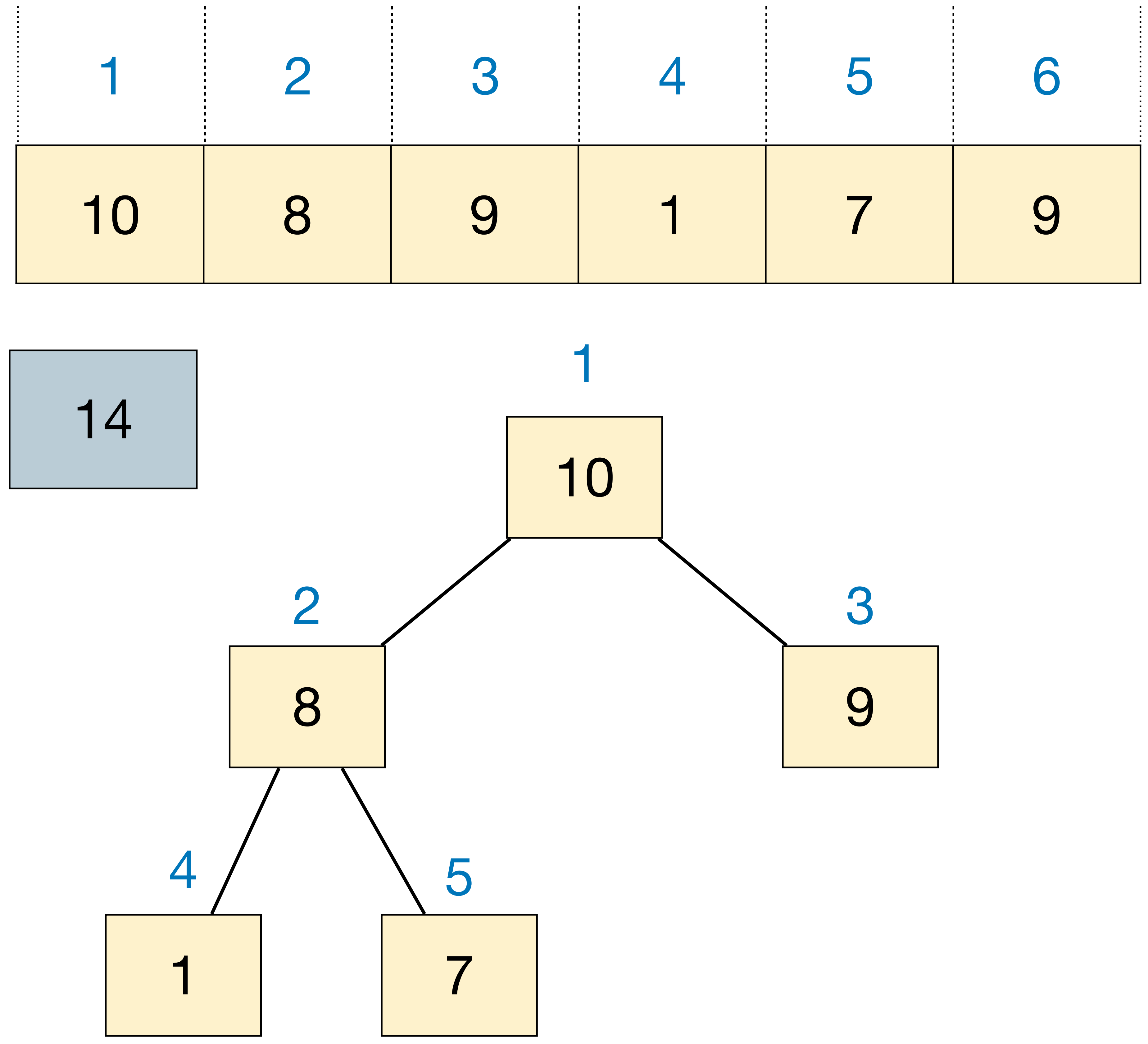

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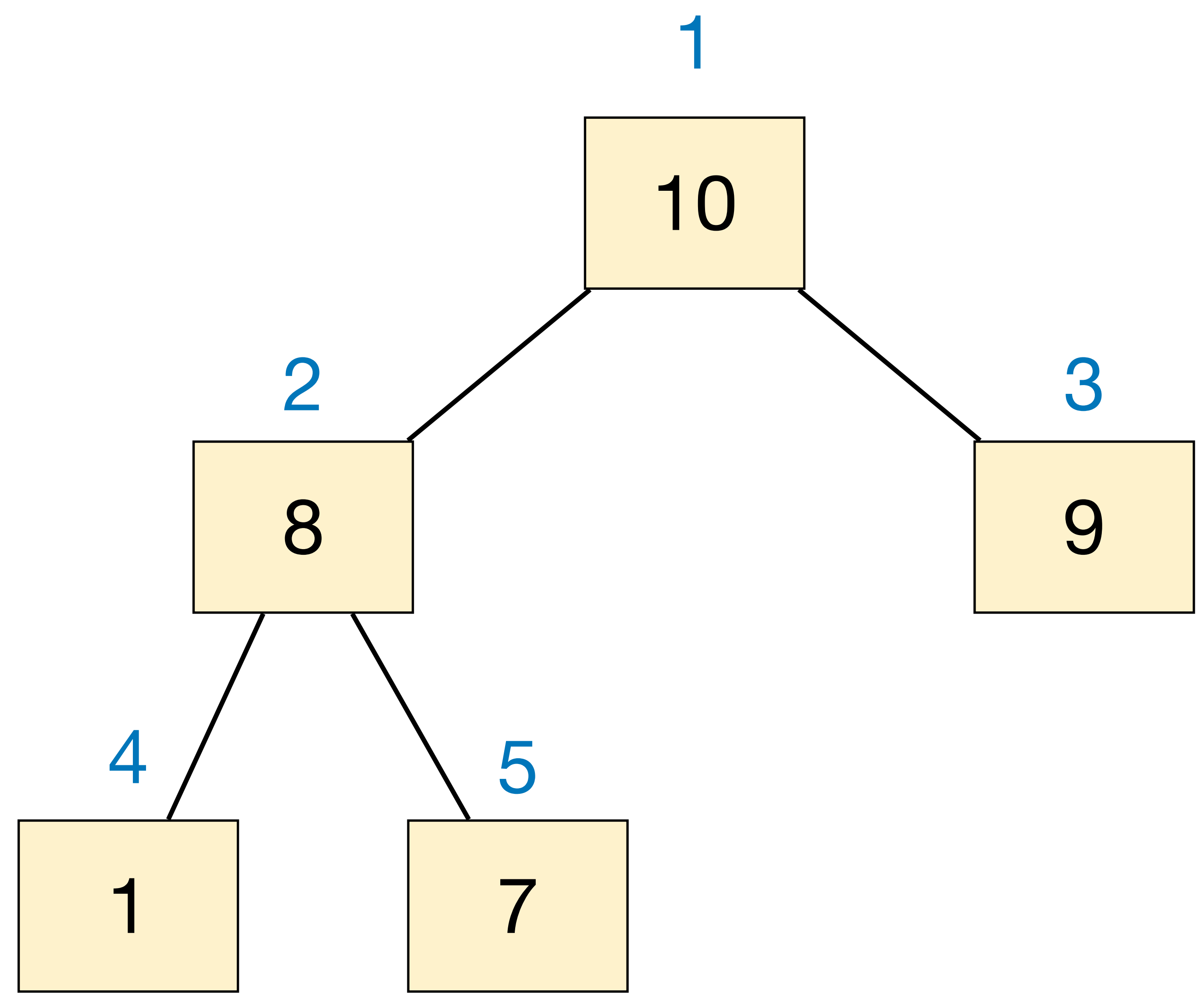

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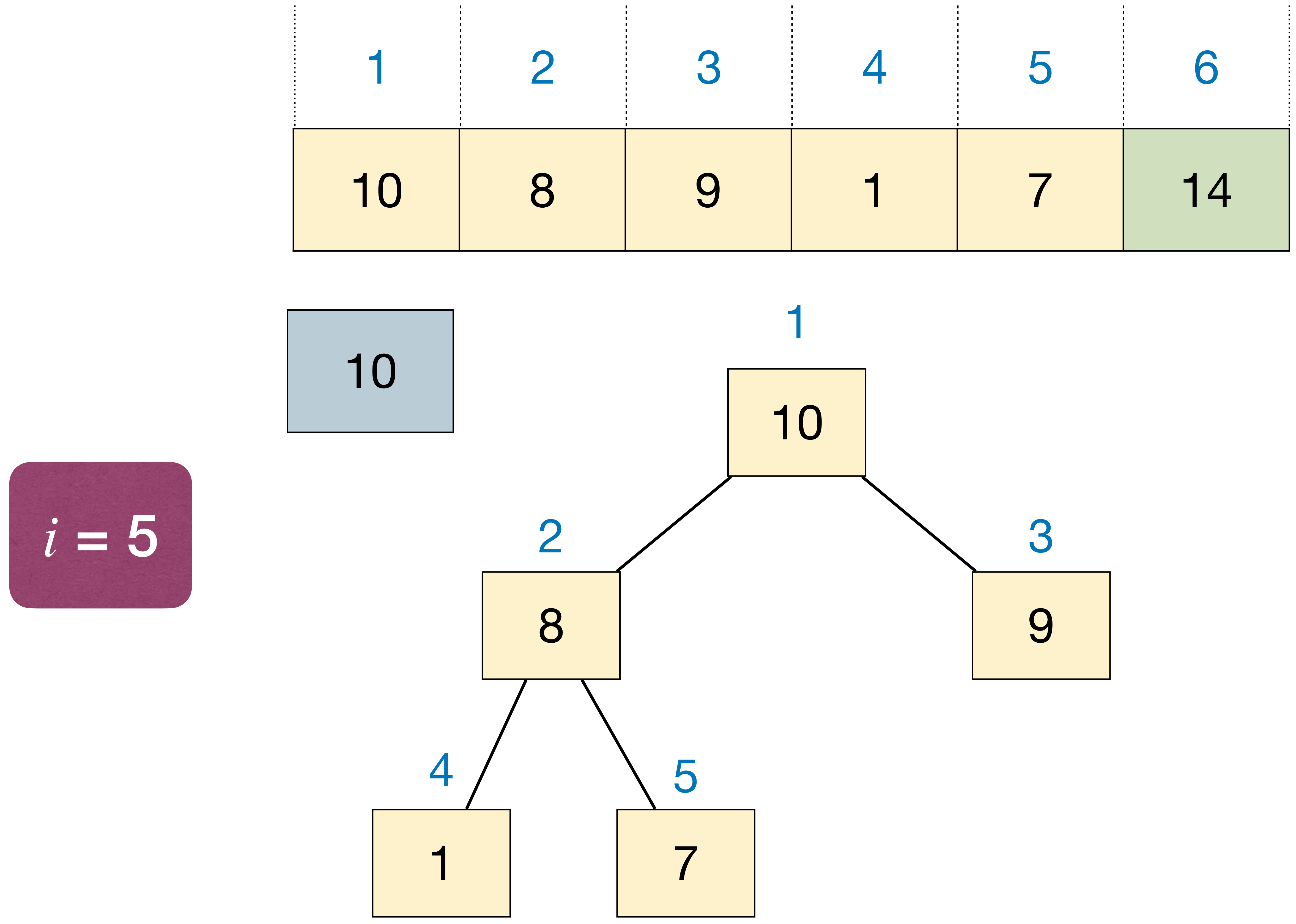

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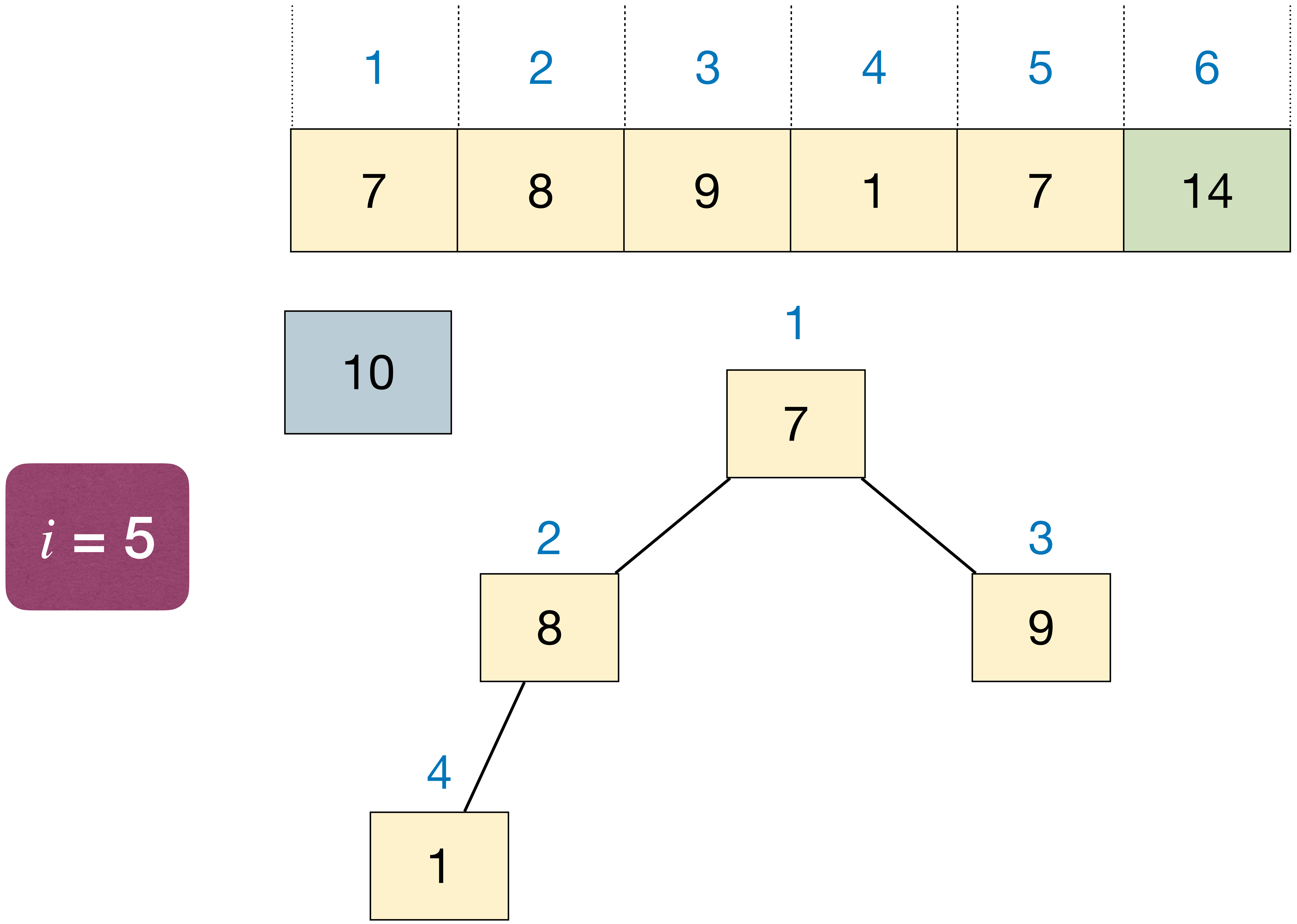

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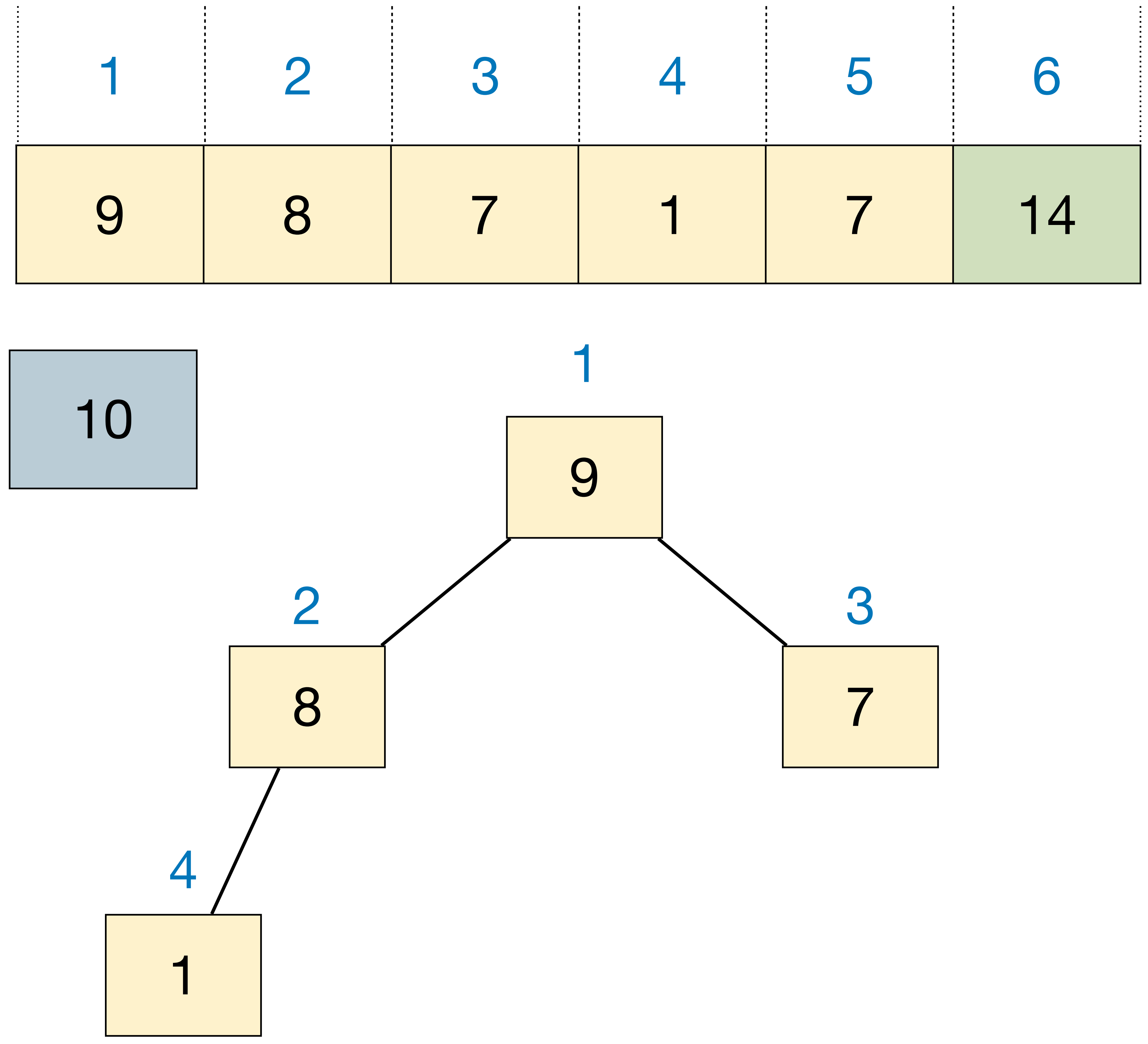

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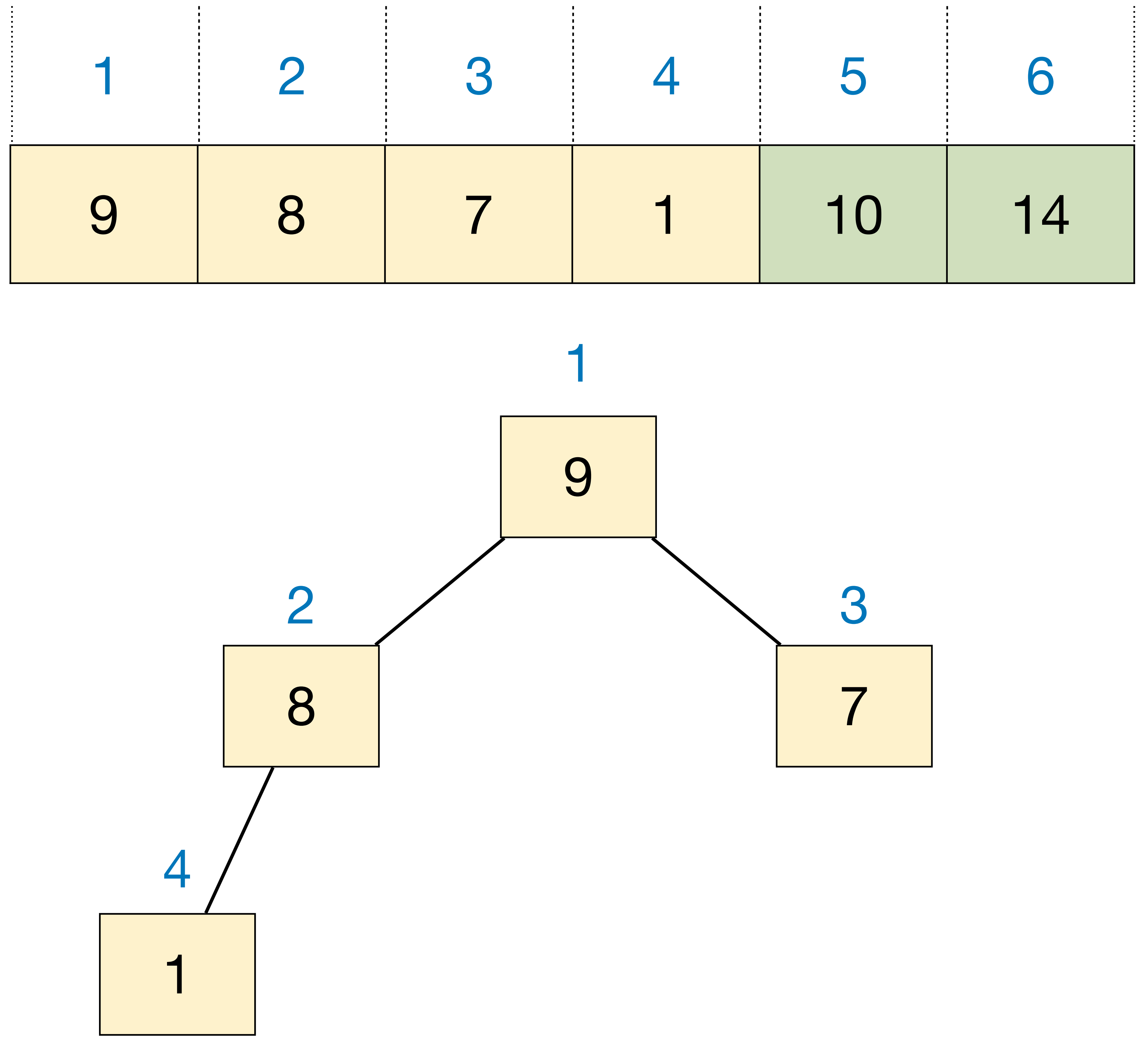

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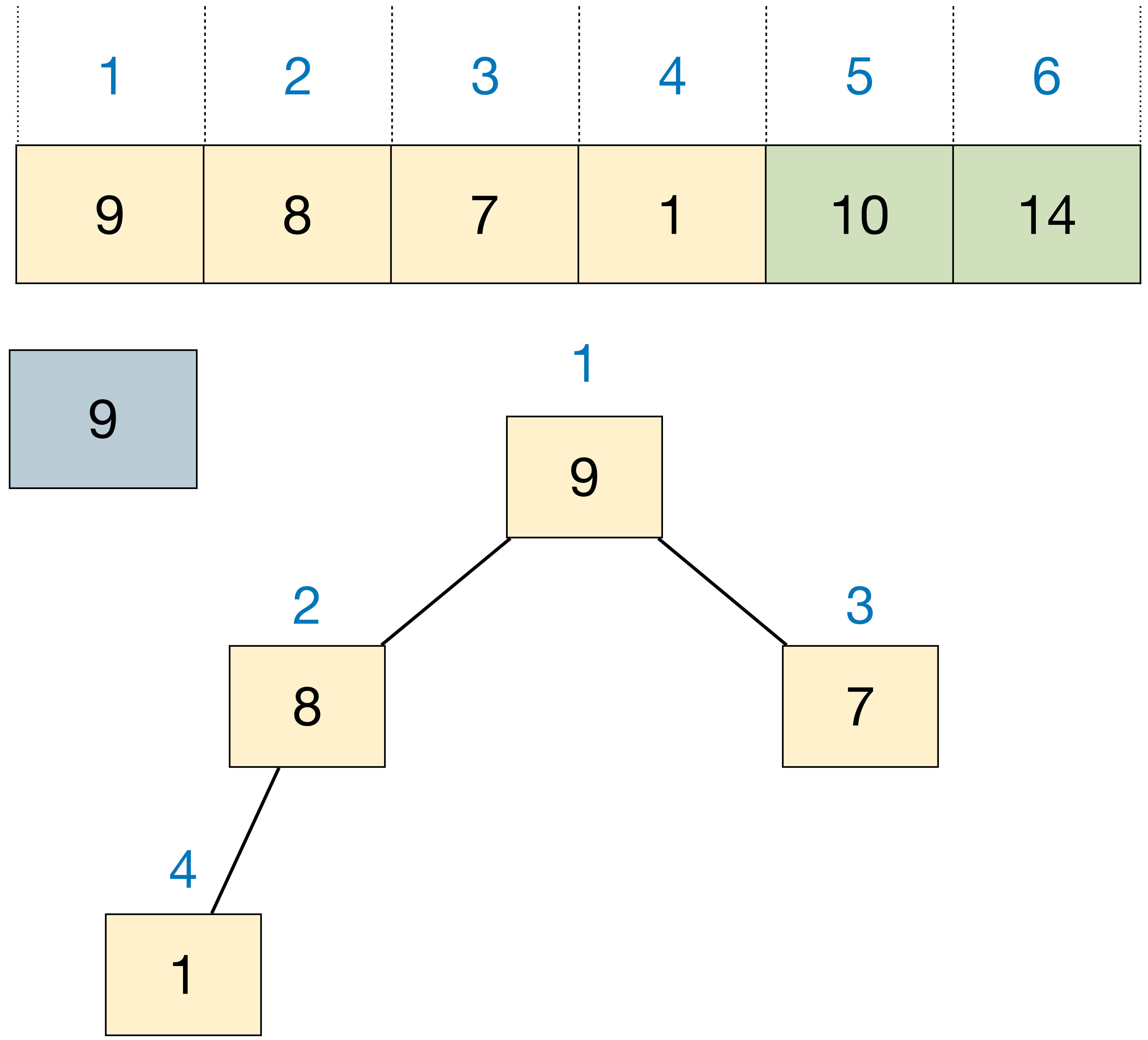

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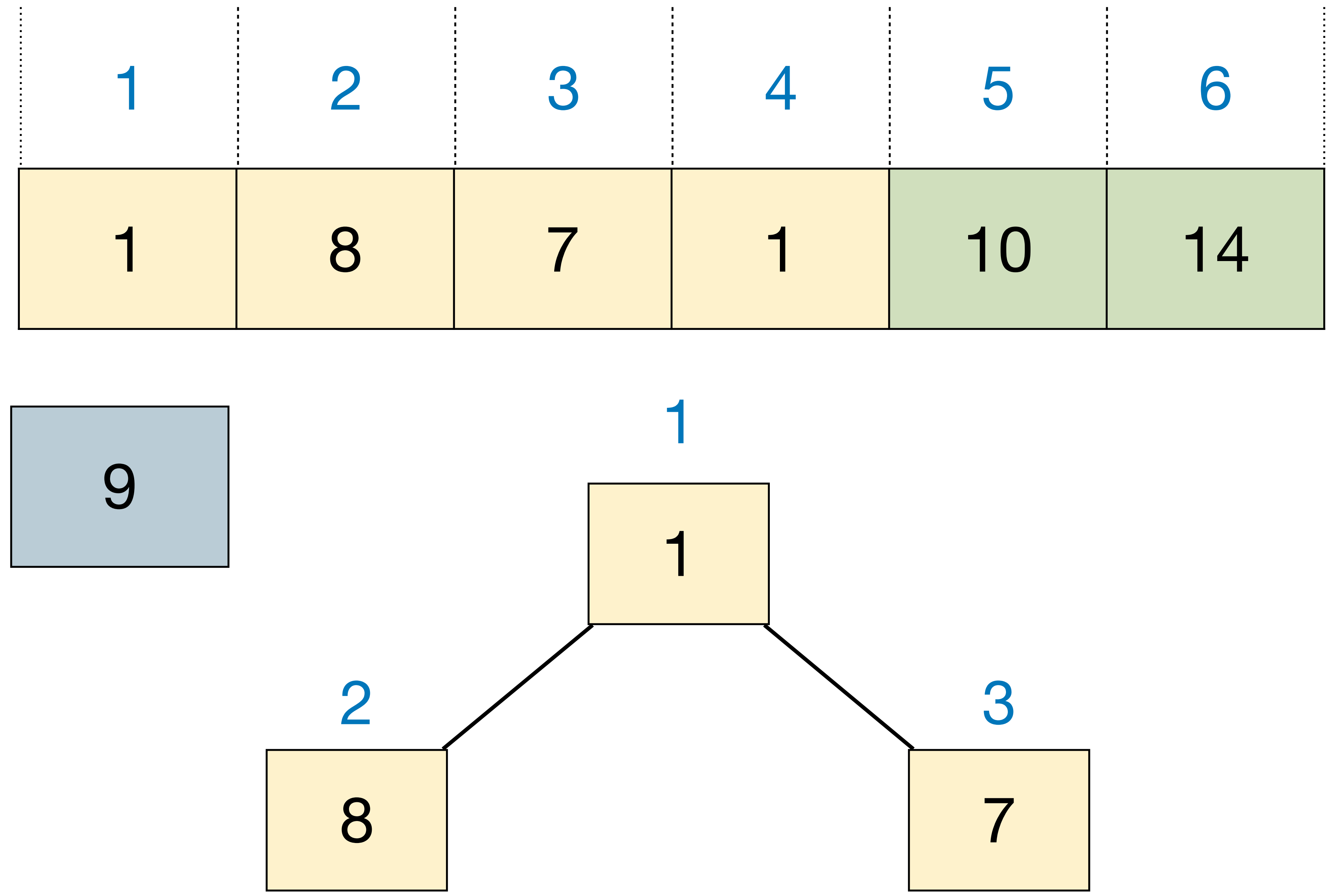

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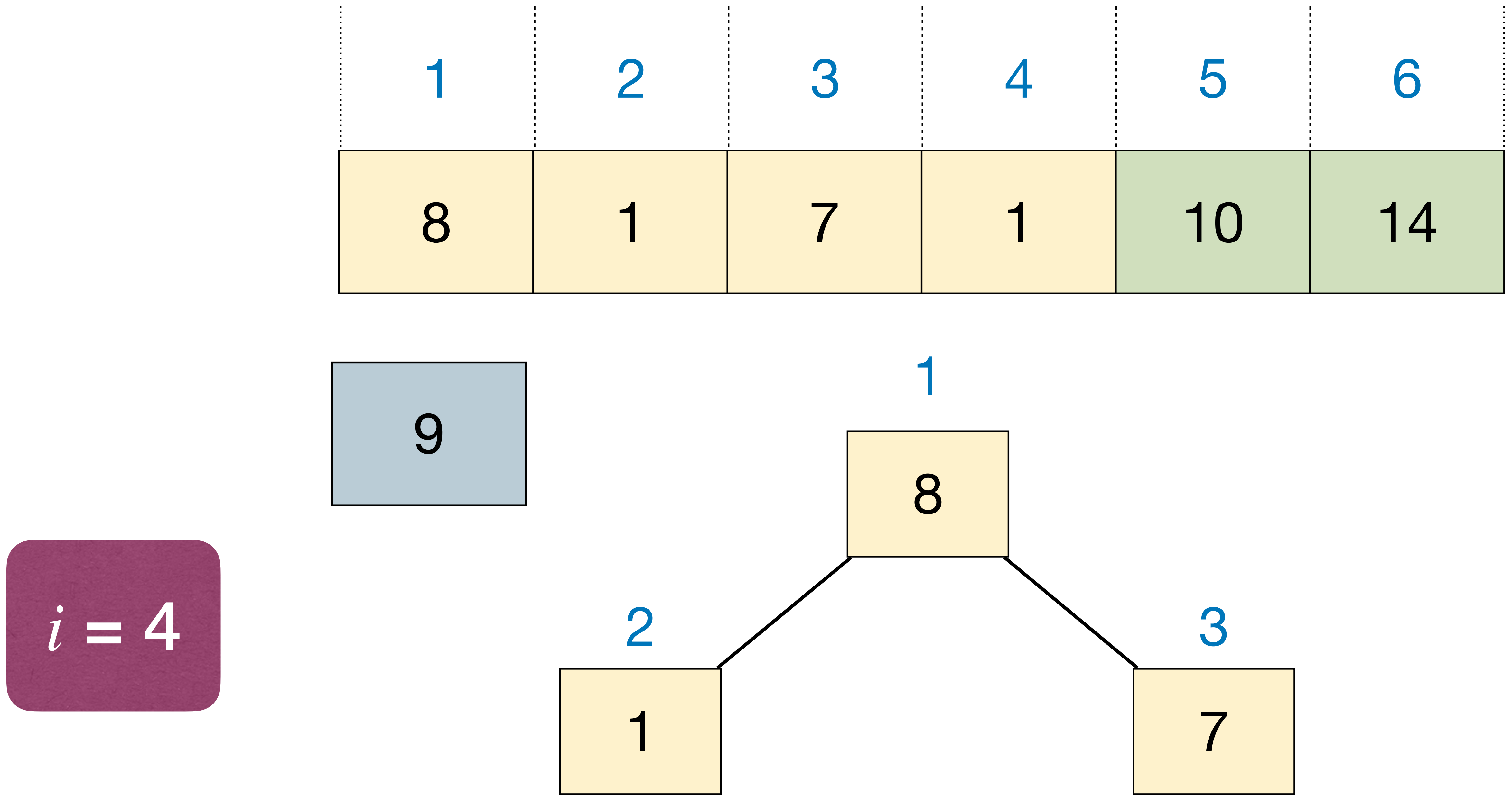

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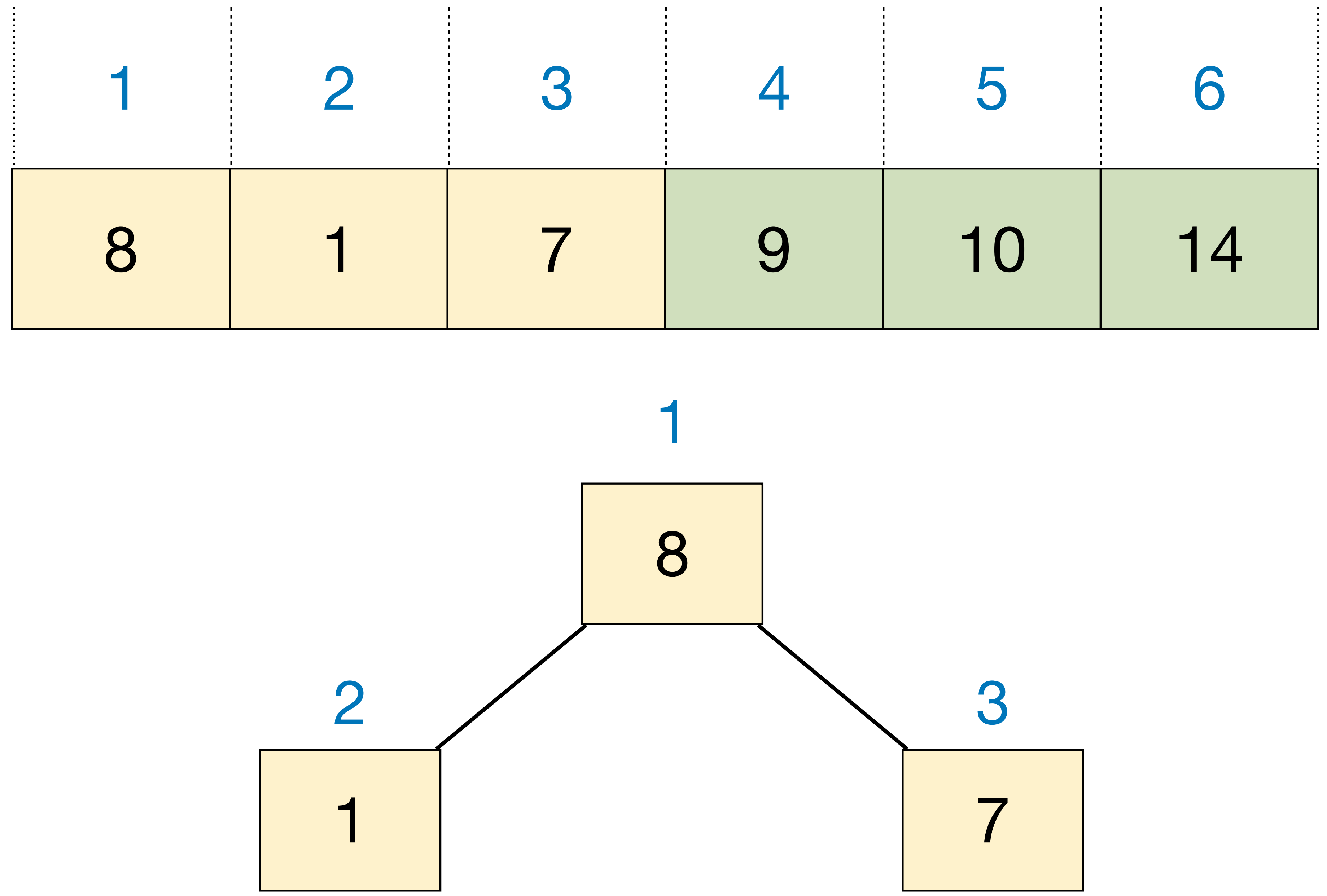

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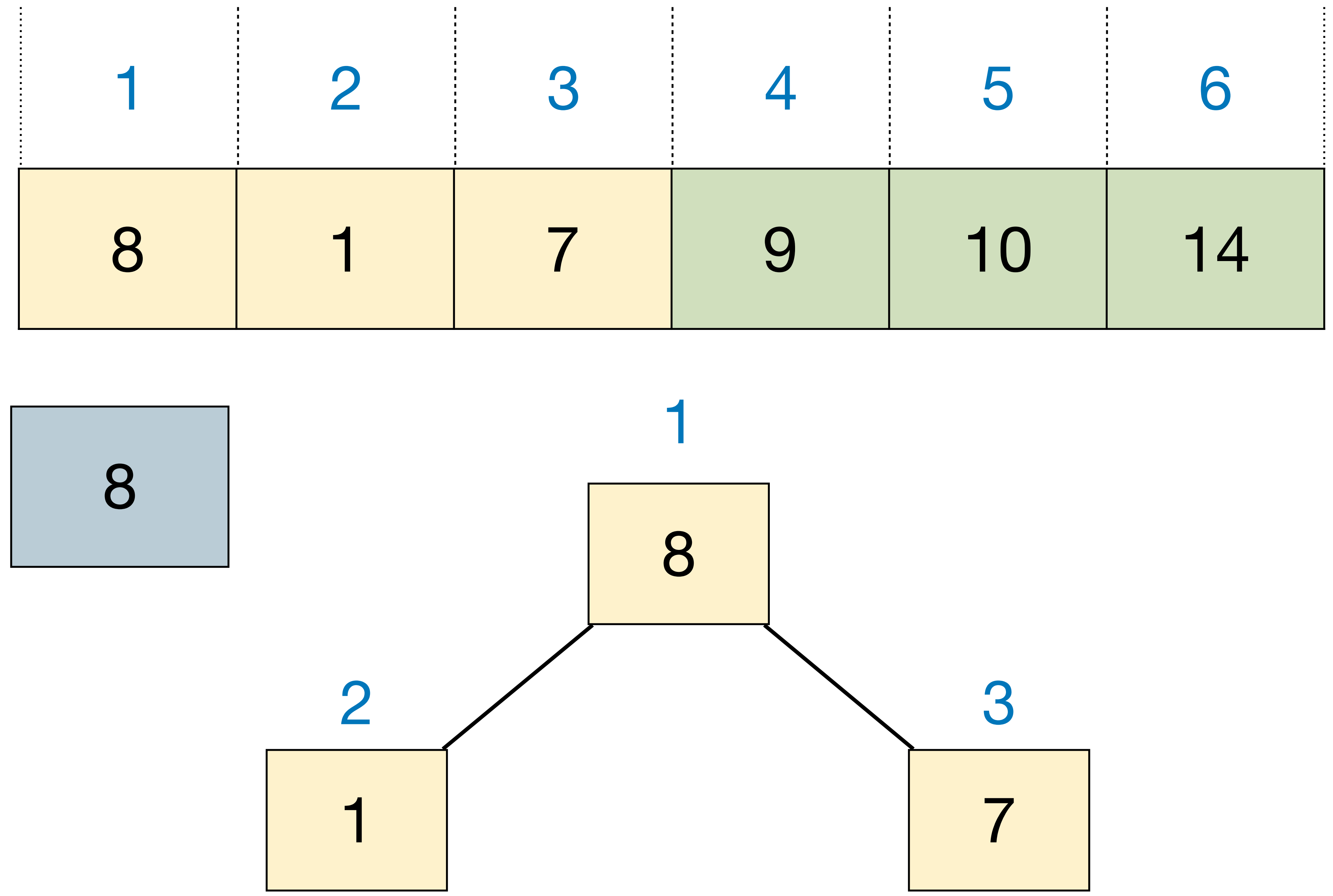

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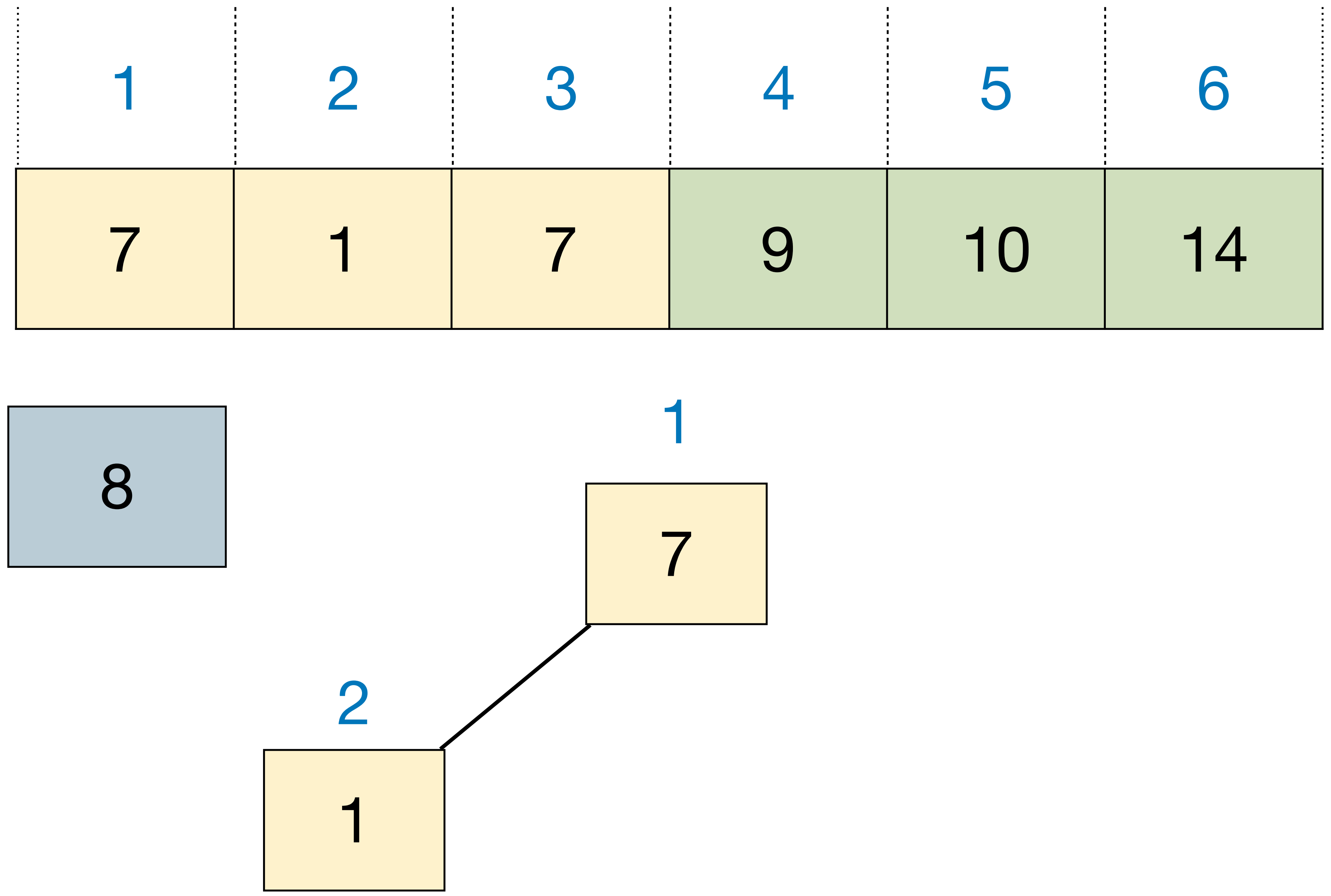

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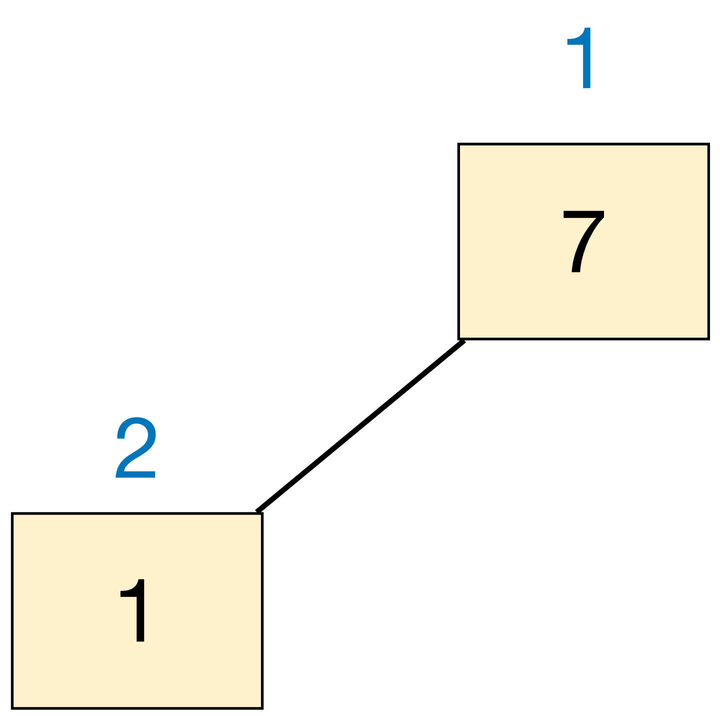

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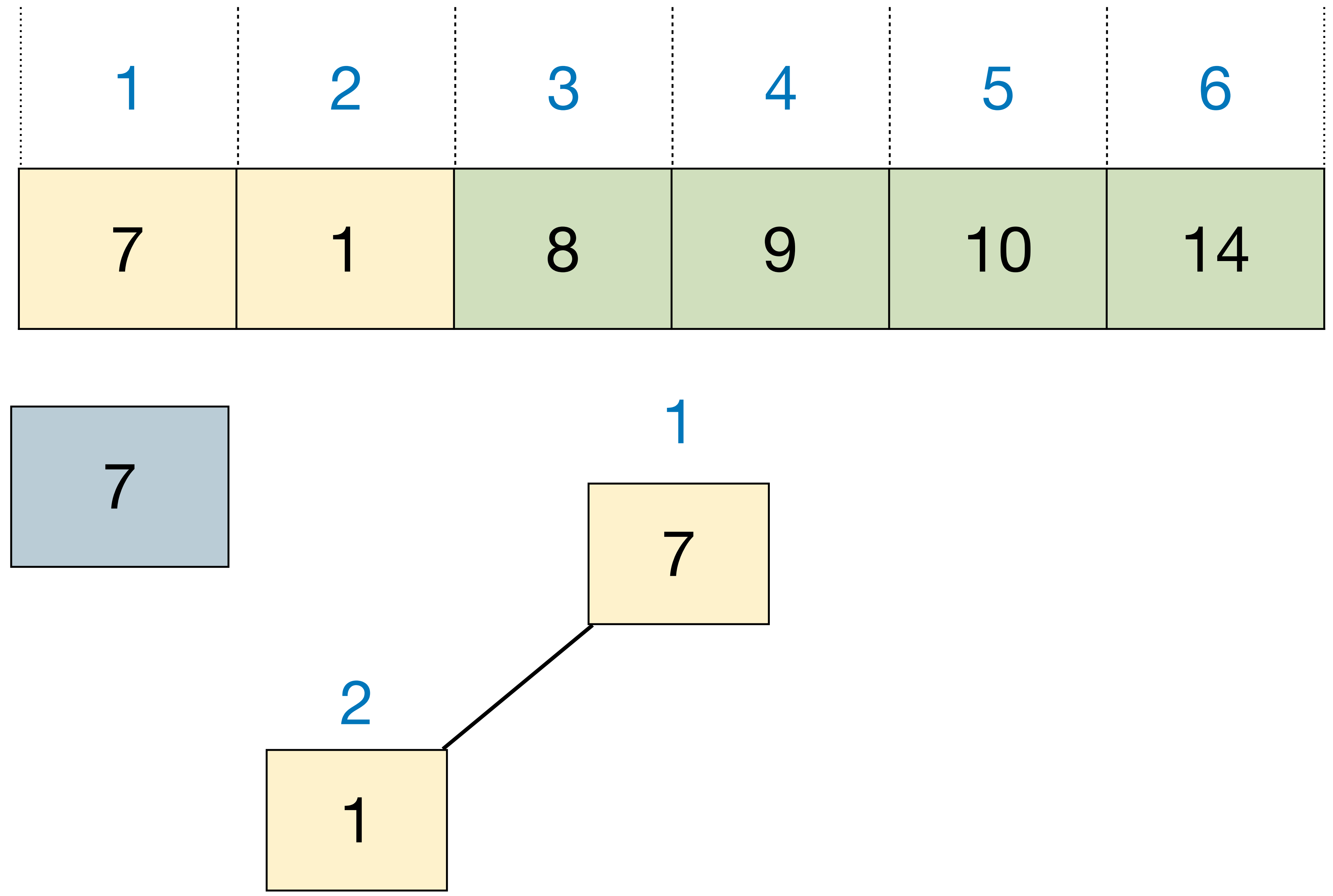
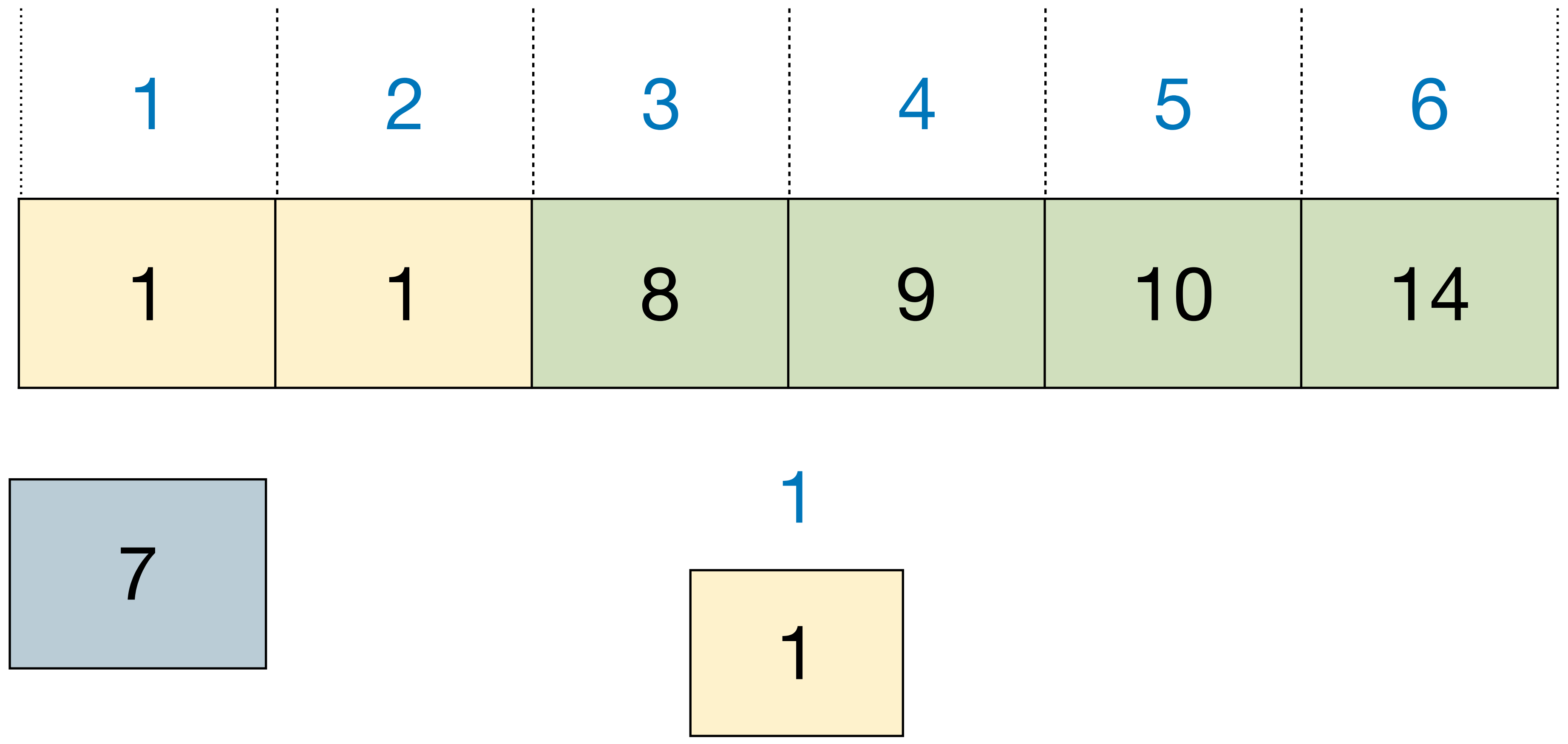
$i = 2$



HeapSort



HeapSort(I):

heap := *BuildMaxHeap*(*I*)

for i := n **down to** 2

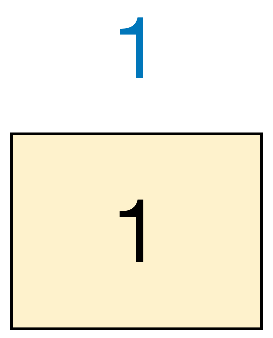
    cur_max := *heap.HeapExtractMax*()

    $I[i]$:= *cur_max*

$i = 2$

HeapSort

1	2	3	4	5	6
1	7	8	9	10	14



HeapSort(I):

$heap := BuildMaxHeap(I)$

for $i := n$ **down to** 2

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  $I[i] := cur\_max$

$i = 2$

HeapSort

HeapSort(I):

$heap := BuildMaxHeap(I)$
for $i := n$ **down to** 2
    $cur\_max := heap.HeapExtractMax()$
    $I[i] := cur\_max$

1. Keep a copy of the root item
2. Remove last item and put it to root
3. Maintain heap property
4. Return the copy of the root item

- Total runtime of these iterations
  - $\sum_{i=2}^{n} O(\lg i) = O(\lg(n!)) = O(n \lg n)$

Stirling's formula

HeapSort

HeapSort(I):

$heap := BuildMaxHeap(I)$

for $i := n$ **down to** 2

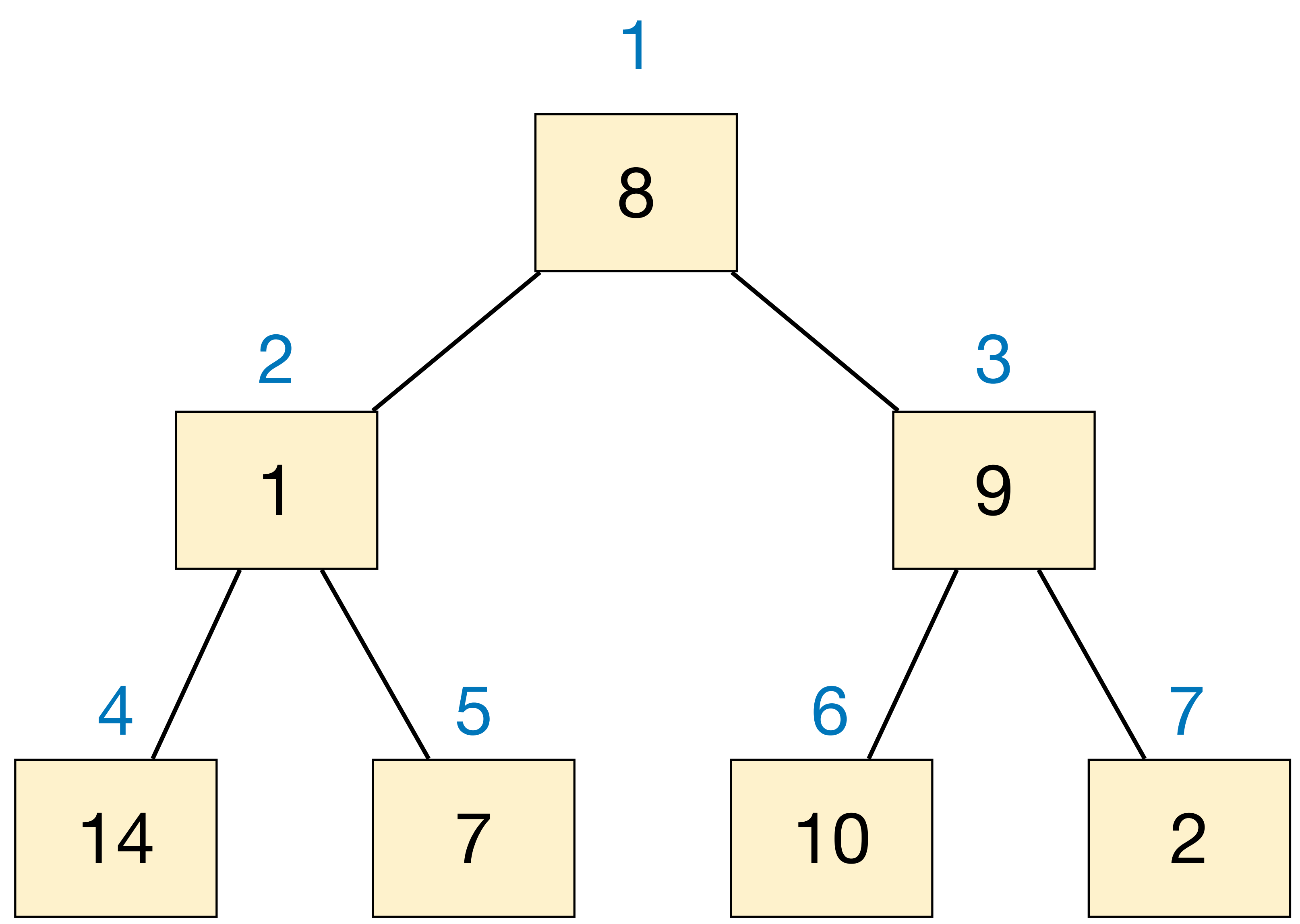
 $cur\_max := heap.HeapExtractMax()$

 $I[i] := cur\_max$

- Given an array $I[1\ldots n]$, how to build a max-heap?
  - Start with an empty heap, then call `HeapInsert` n times?
  - Cost is $\sum_{i=1}^{n} O(\lg i) = O(n \lg n)$
  - Not bad, but we can do better.

HeapSort

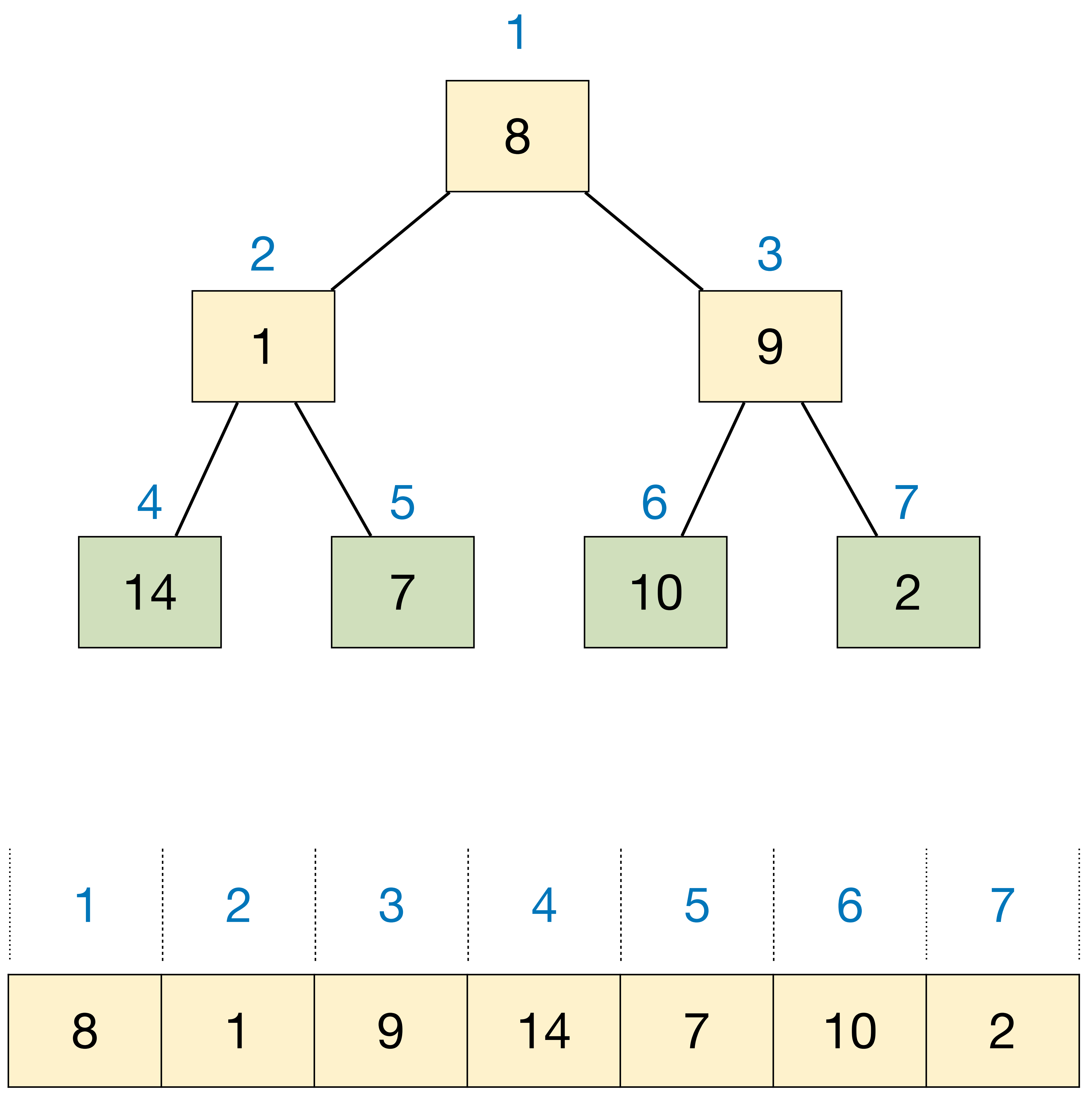
- Given an array $I[1 \ldots n]$, how to build a max-heap?
  - Bottom-up approach: keep merging small heaps into larger ones, until a single heap remains.



1	2	3	4	5	6	7
8	1	9	14	7	10	2

HeapSort

- Given an array $I[1\ldots n]$, how to build a max-heap?
  - Bottom-up approach: keep merging small heaps into larger ones, until a single heap remains.
  - Each leaf node is a 1-item heap.



HeapSort

- Given an array $I[1 \ldots n]$, how to build a max-heap?
  - Bottom-up approach: keep merging small heaps into larger ones, until a single heap remains.
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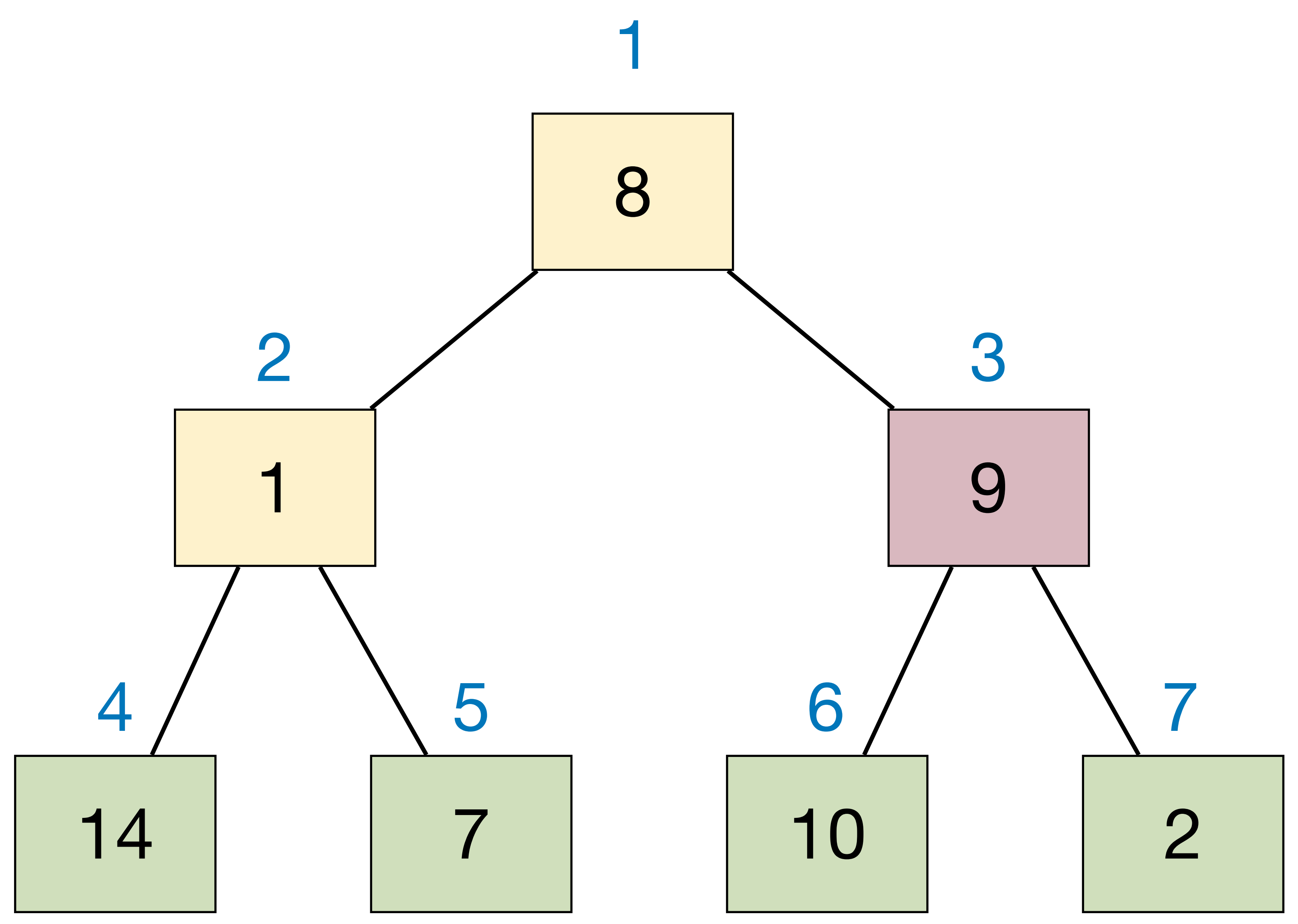
  - Go through remaining nodes in index decreasing order: at each node, we are merging two heaps.



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HeapSort

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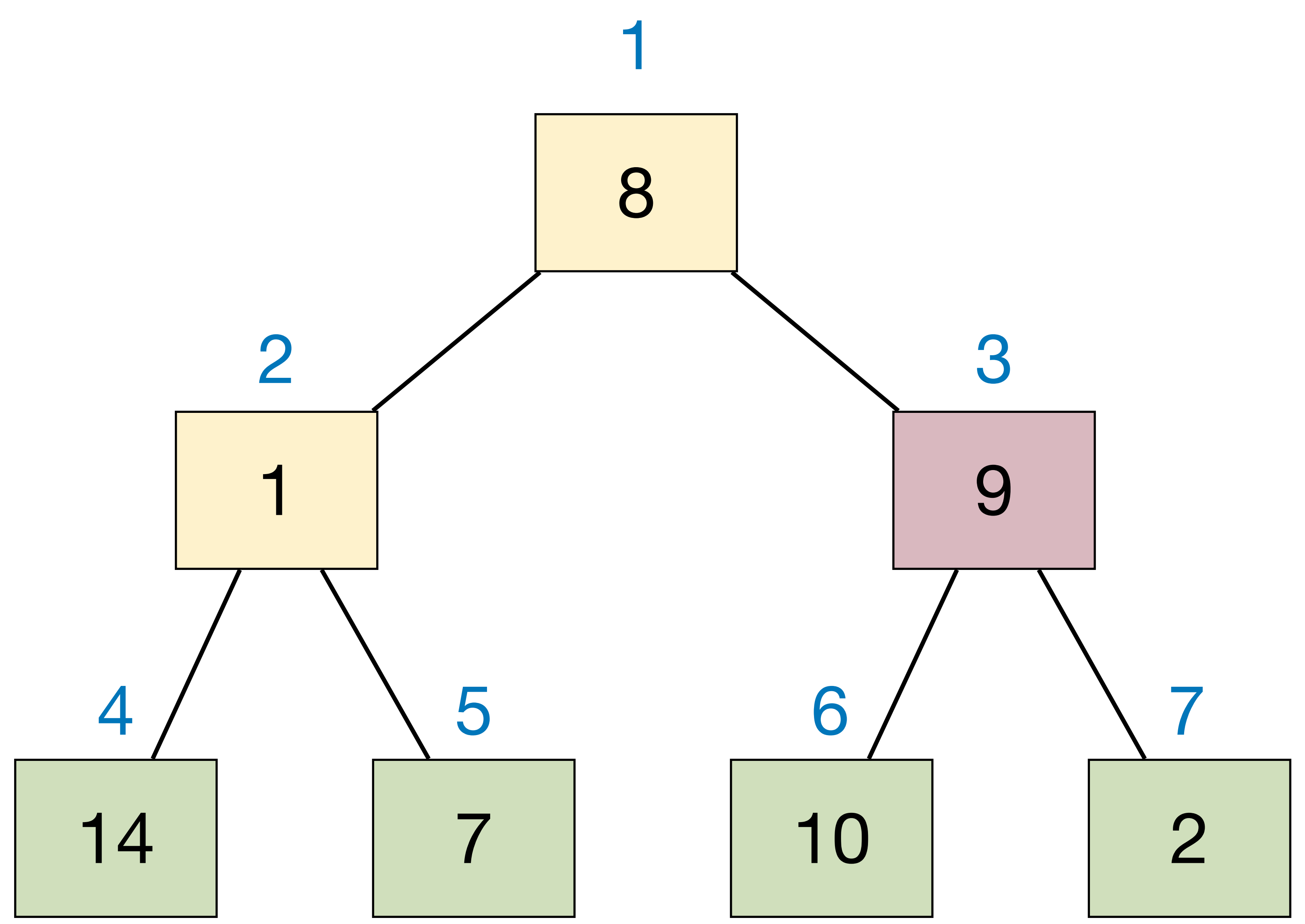
  - Maintain heap property during merging: use `MaxHeapify`.



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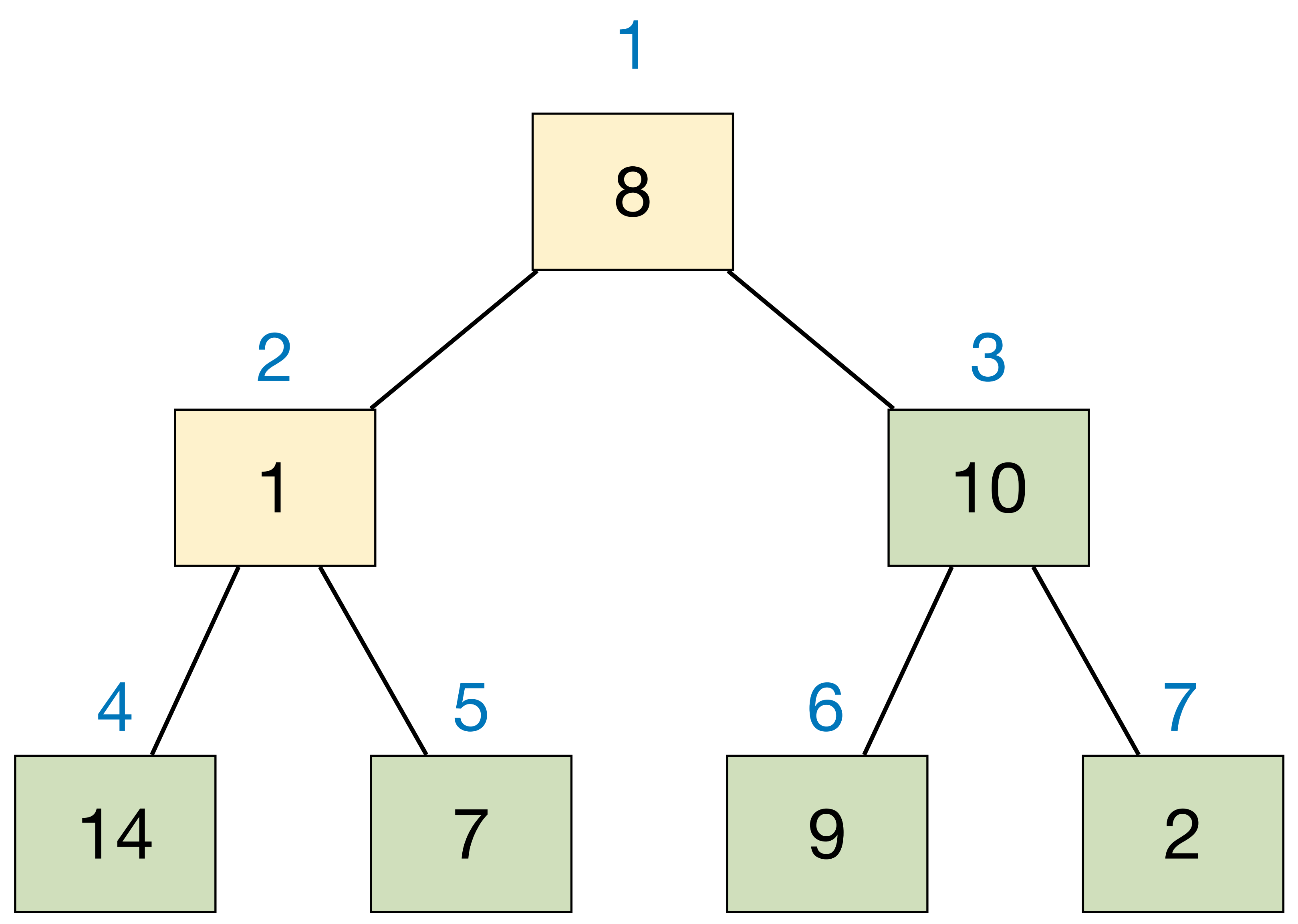

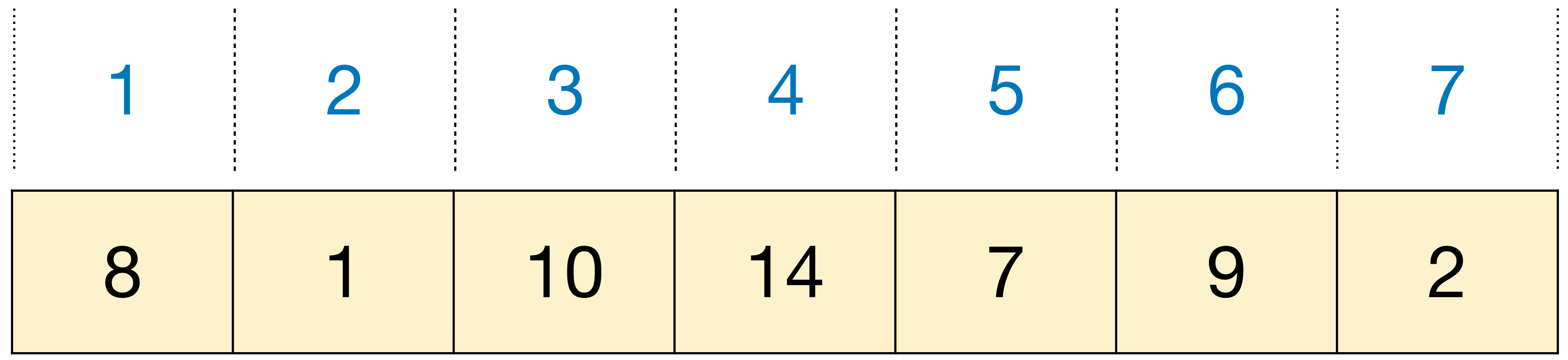

HeapSort

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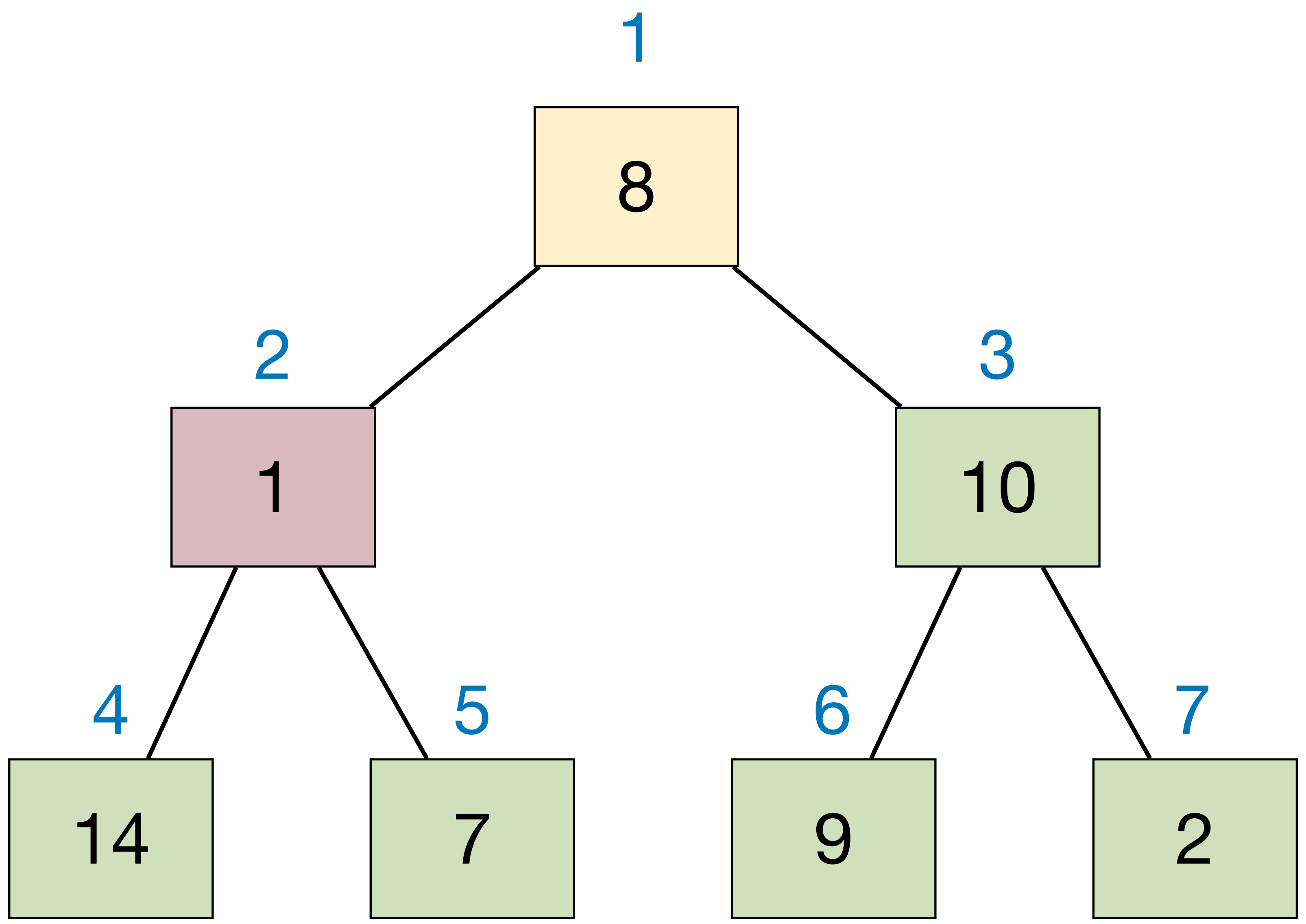

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HeapSort

- Given an array $I[1...n]$, how to build a max-heap?
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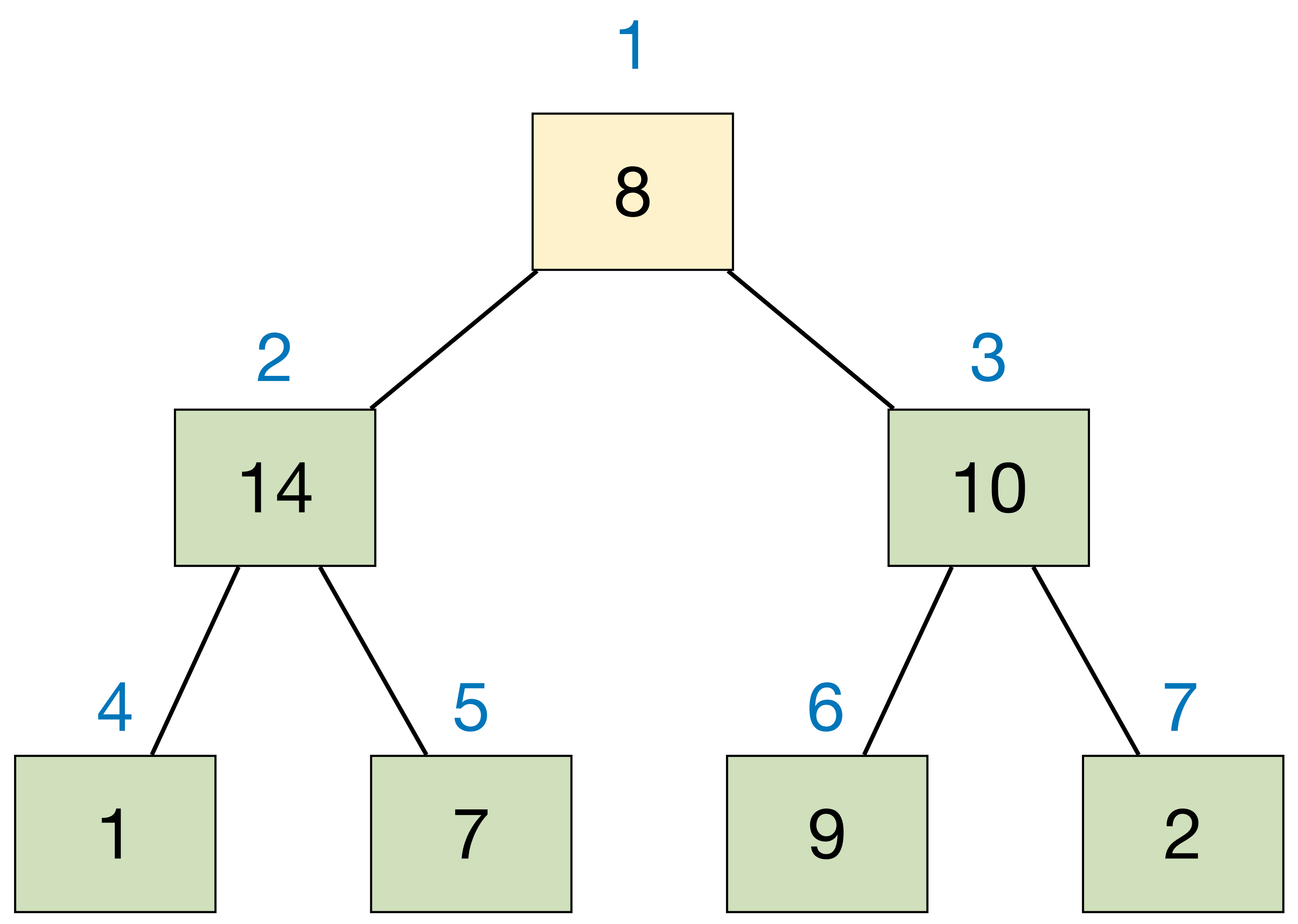

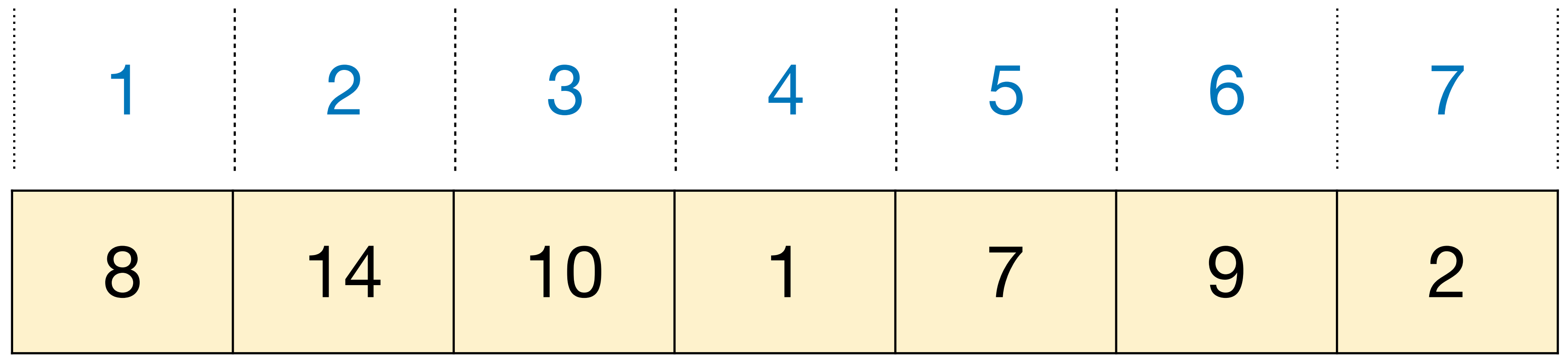

HeapSort

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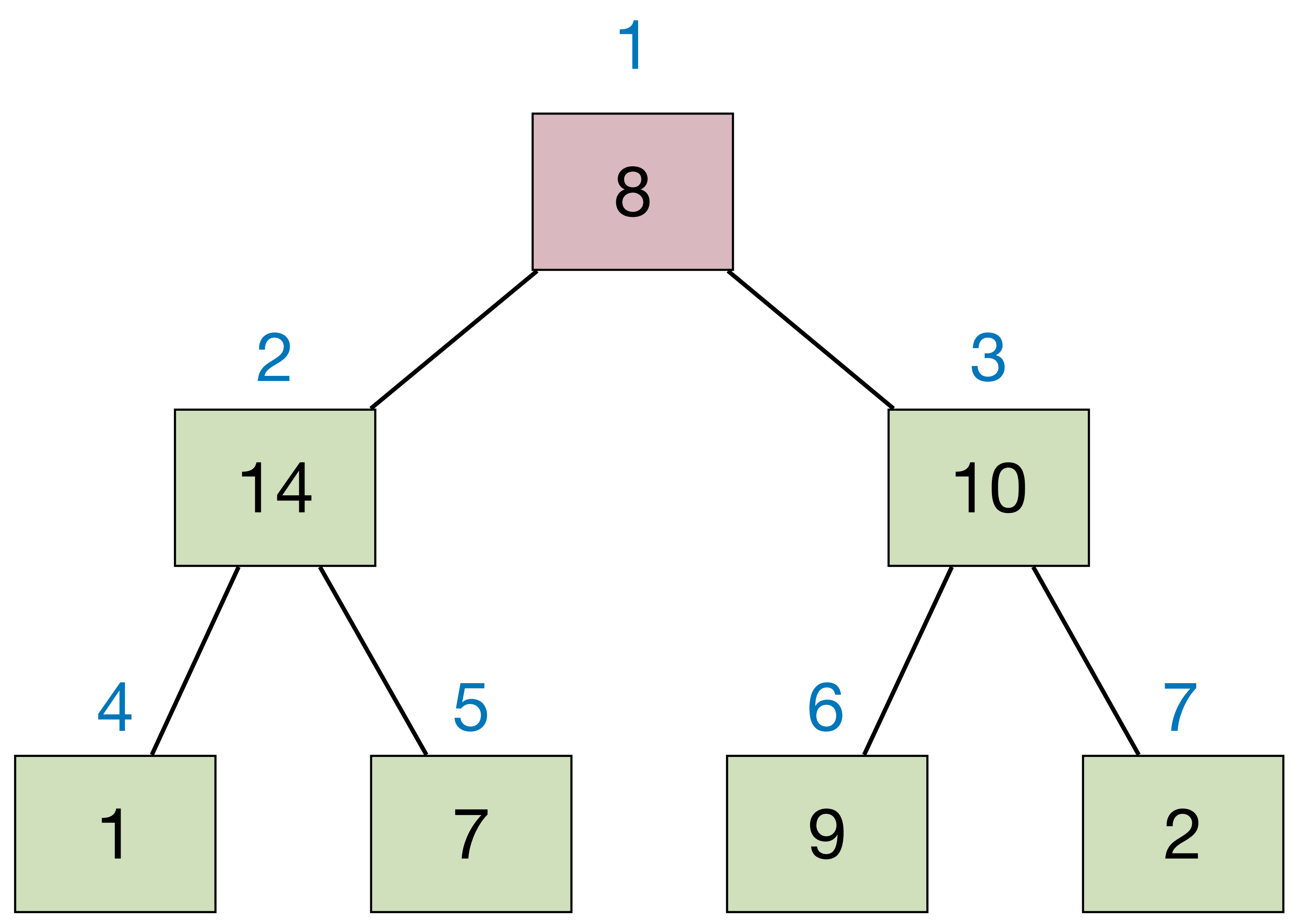

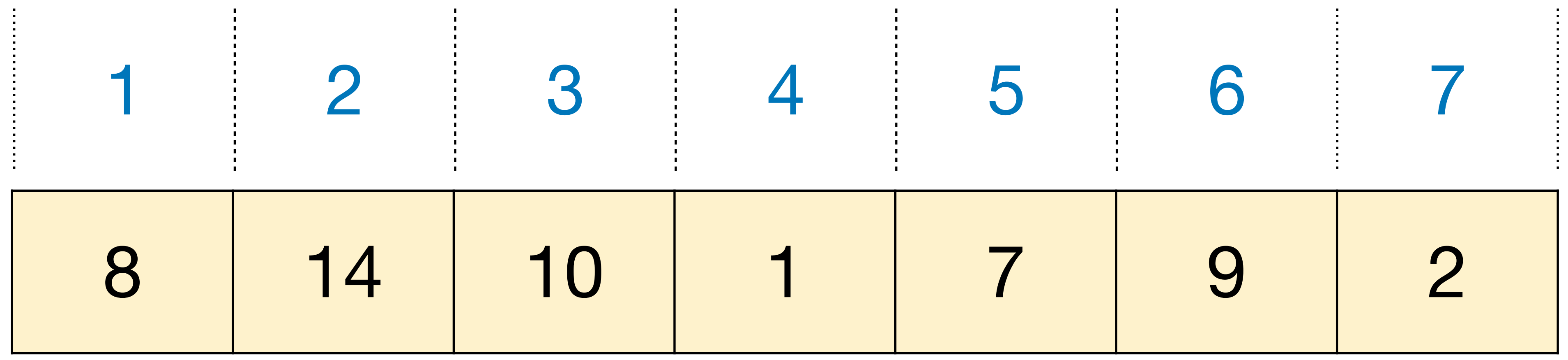

HeapSort

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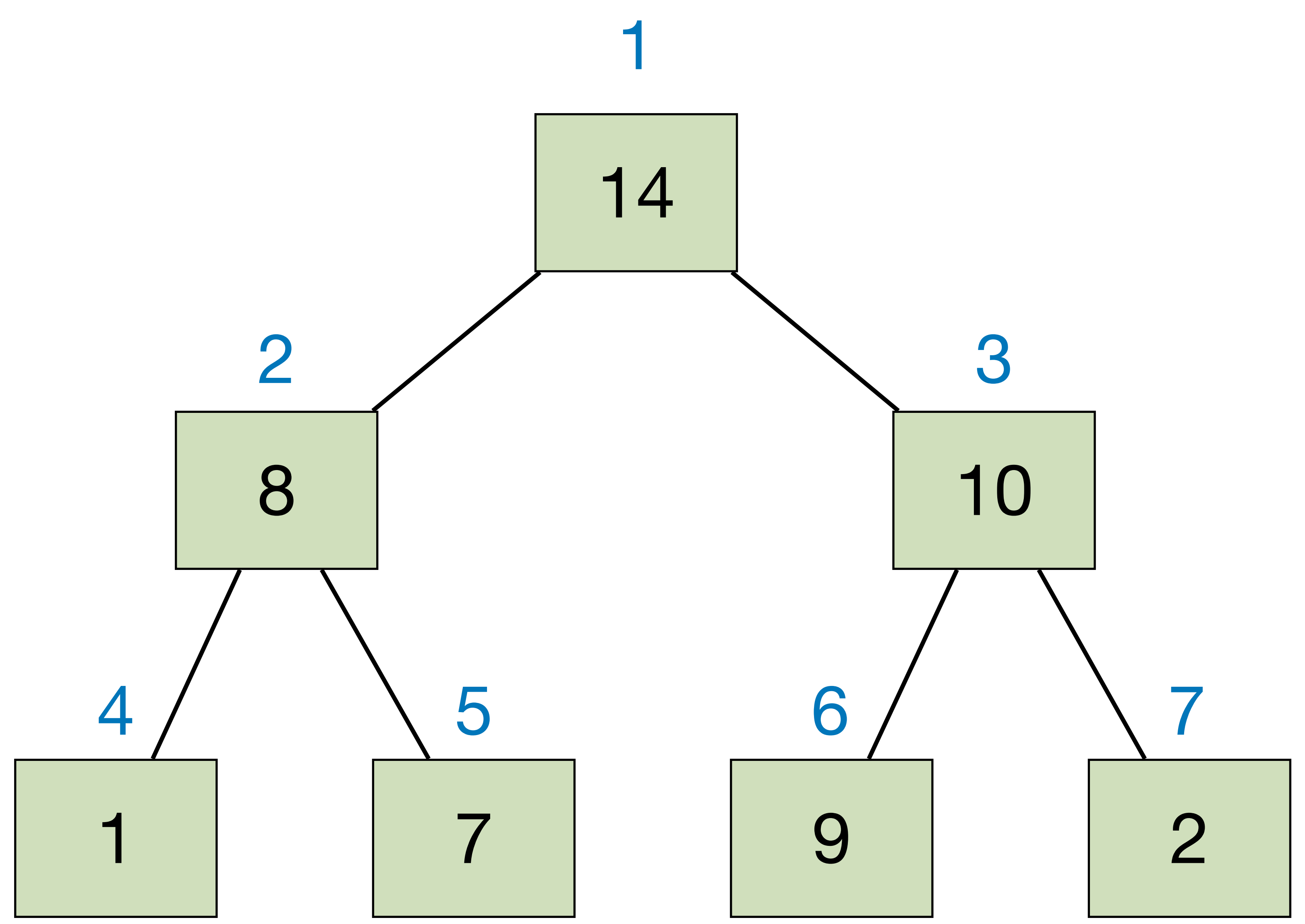


1	2	3	4	5	6	7
14	8	10	1	7	9	2

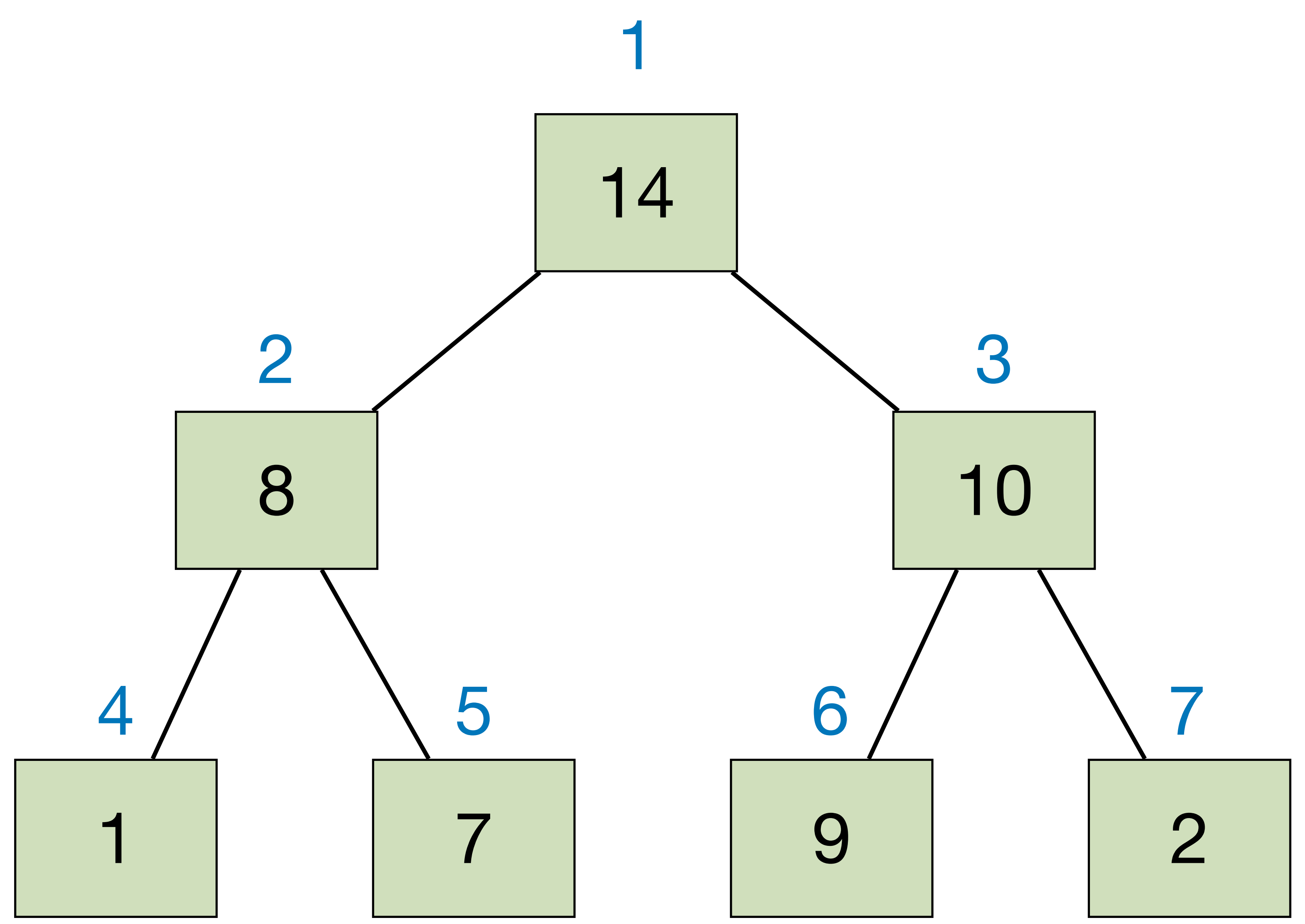
HeapSort

BuildMaxHeap(A):

heap_size := n

for i := *Floor*(n/2) **down to** 1

    MaxHeapify(i, A)



1	2	3	4	5	6	7
14	8	10	1	7	9	2

- Time complexity of `BuildMaxHeap`?
  - $\Theta(n)$ calls to `MaxHeapify`, each costing $O(\lg n)$, so $O(n \lg n)$?
  - Correct but not tight…

HeapSort

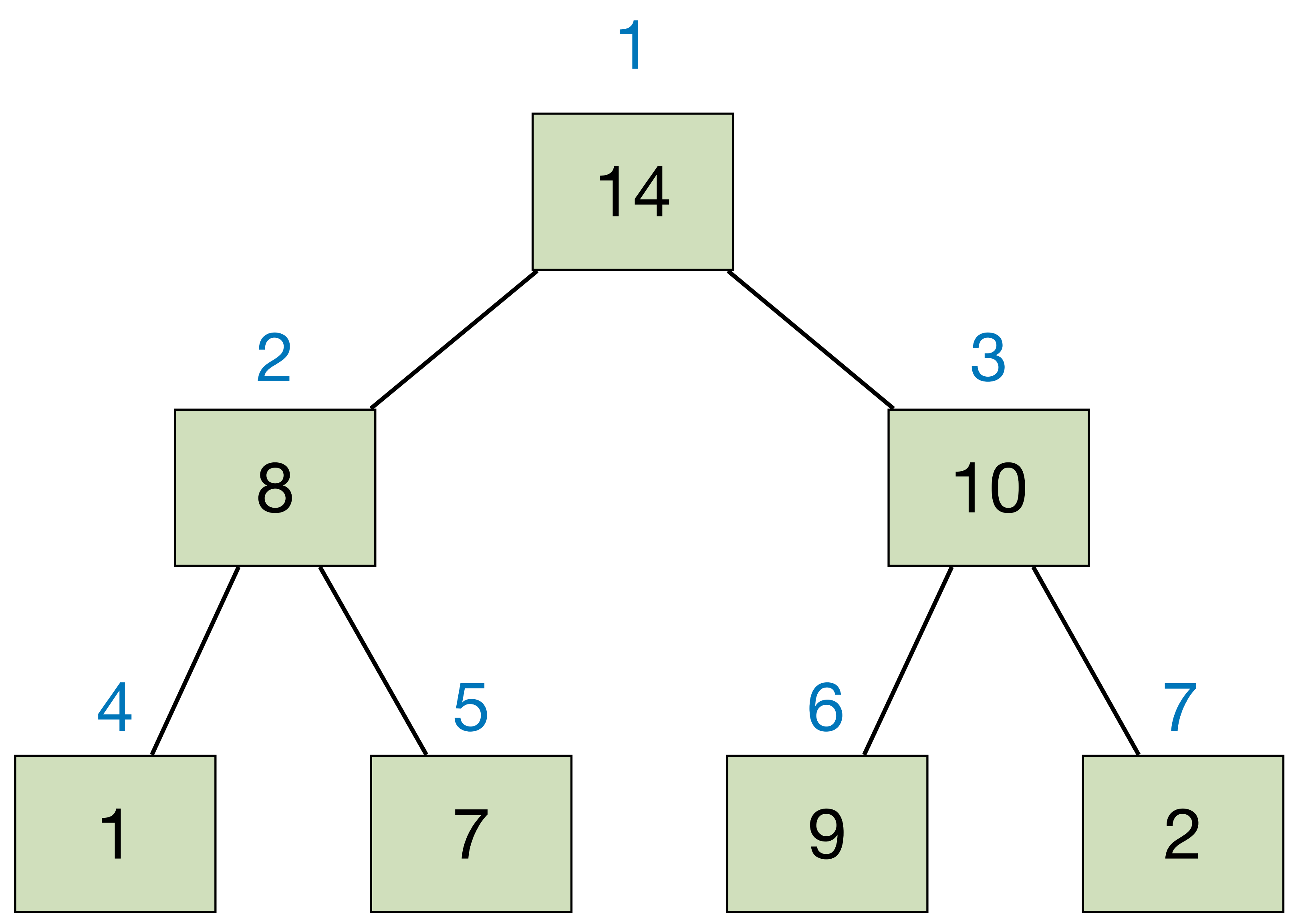
BuildMaxHeap(A):

$heap\_size := n$

for $i := Floor(n/2)$ **down to** 1

  $MaxHeapify(i, A)$

- Height of n-items heap is $\lfloor \lg n \rfloor$
- Any height h has $\leq \lceil \frac{n}{2^{h+1}} \rceil$ nodes
- Cost of all MaxHeapify：
  - $$\sum_{h=0}^{\lfloor \lg n \rfloor} (\lceil \frac{n}{2^{h+1}} \rceil \cdot O(h)) = O(n \cdot \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h}) = O(n)$$



1	2	3	4	5	6	7
14	8	10	1	7	9	2

HeapSort

HeapSort(I):

$heap := BuildMaxHeap(I)$

for $i := n$ **down to** 2

    $cur\_max := heap.HeapExtractMax()$

    I[i] $:= cur\_max$

BuildMaxHeap(A):

$heap\_size := n$

for $i := Floor(n/2)$ **down to** 1

    $MaxHeapify(i, A)$

Time Complexity: $O(n)$

Time Complexity: $O(n \lg n)$

- Time complexity of `HeapSort` is $O(n \lg n)$.
- Extra space required during execution is $O(1)$.

Further reading

- [CLRS] Ch.6