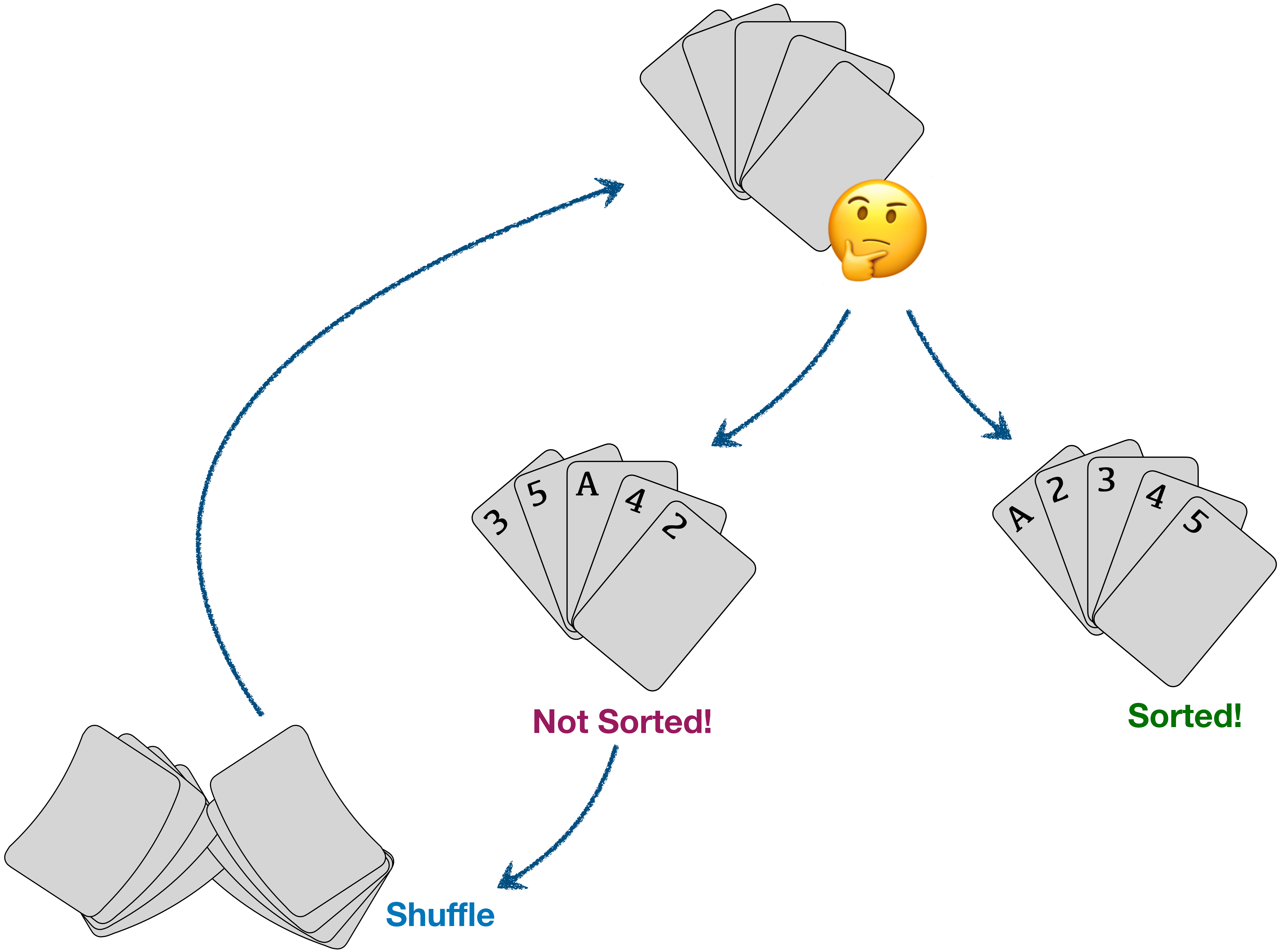
排序
Sorting

钮鑫涛
Nanjing University
2024 Fall

The slides are mainly adapted from the original ones shared by Chaodong Zheng and Kevin Wayne. Thanks for their supports! We also use some materials from stanford-cs161.

Bogosort: The stupid sort



My Bogosort, when it doesn't sort the array correctly the thousands time

The Sorting Problem

- Sort n numbers into ascending order.
- We can actually sort a collection of any type of data, as long as a **total order** is defined for that type of data.
- That is, for any distinct data items a and b, we compare them, i.e., we can determine:
  - $a < b$, or $b < a$, otherwise, $a = b$, where “<” is a binary relation:
    - E.g., in Java, to use `Collections.sort(List<DataType> list, Comparator<DataType> comparator)` for sorting, you should implement the `comparator` and define the following function in it:

```
public int compare(DataType item1, DataType item2)
```

- We can also sort partially ordered items (more on this later).

Sorting algorithms till now

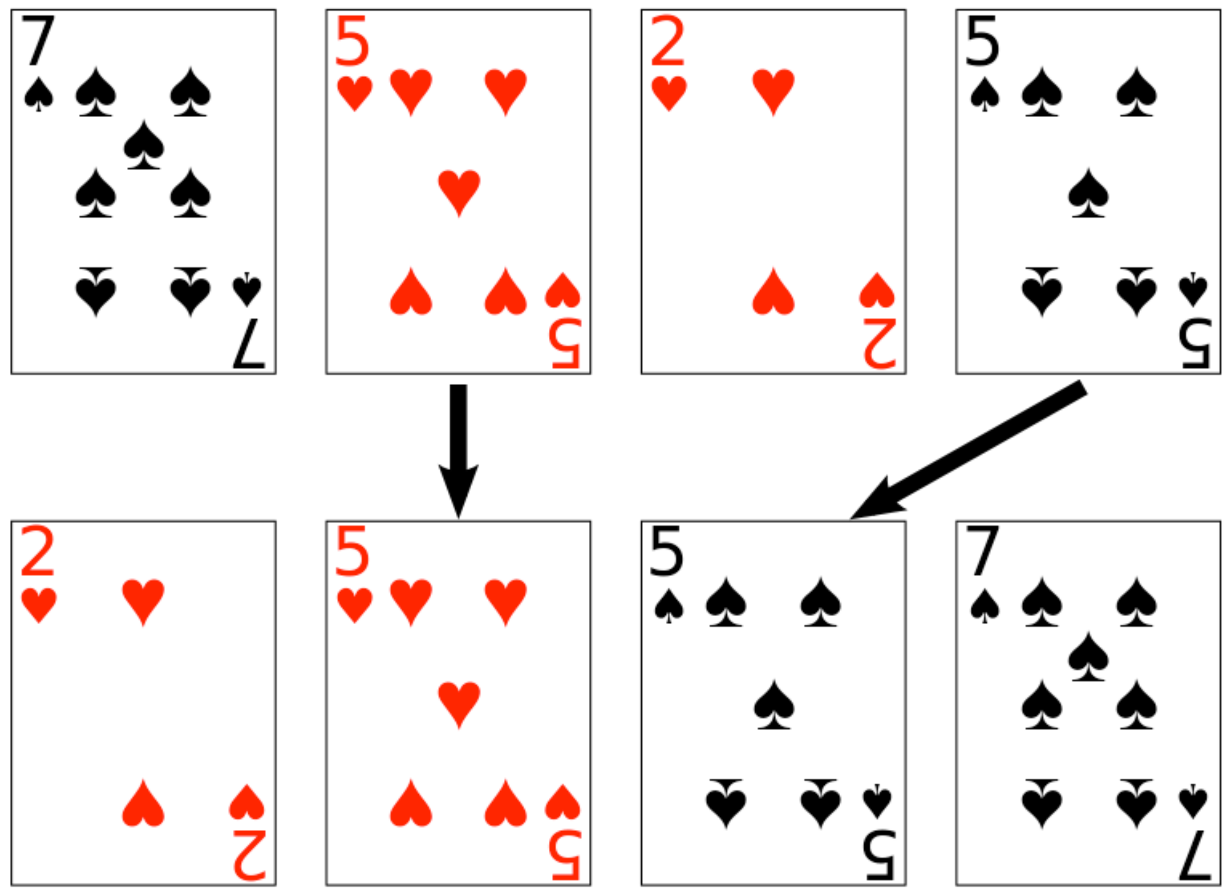
- **Insertion Sort**: gradually increase size of sorted part.
  - $O(n^2)$ time, $O(1)$ space
- **Merge Sort**: example of divide-and-conquer
  - $O(n \log n)$ time, $O(n)$ space
- **Heap Sort**: leverage the heap data structure
  - $O(n \log n)$ time, $O(1)$ space

Characteristics of sorting algorithms

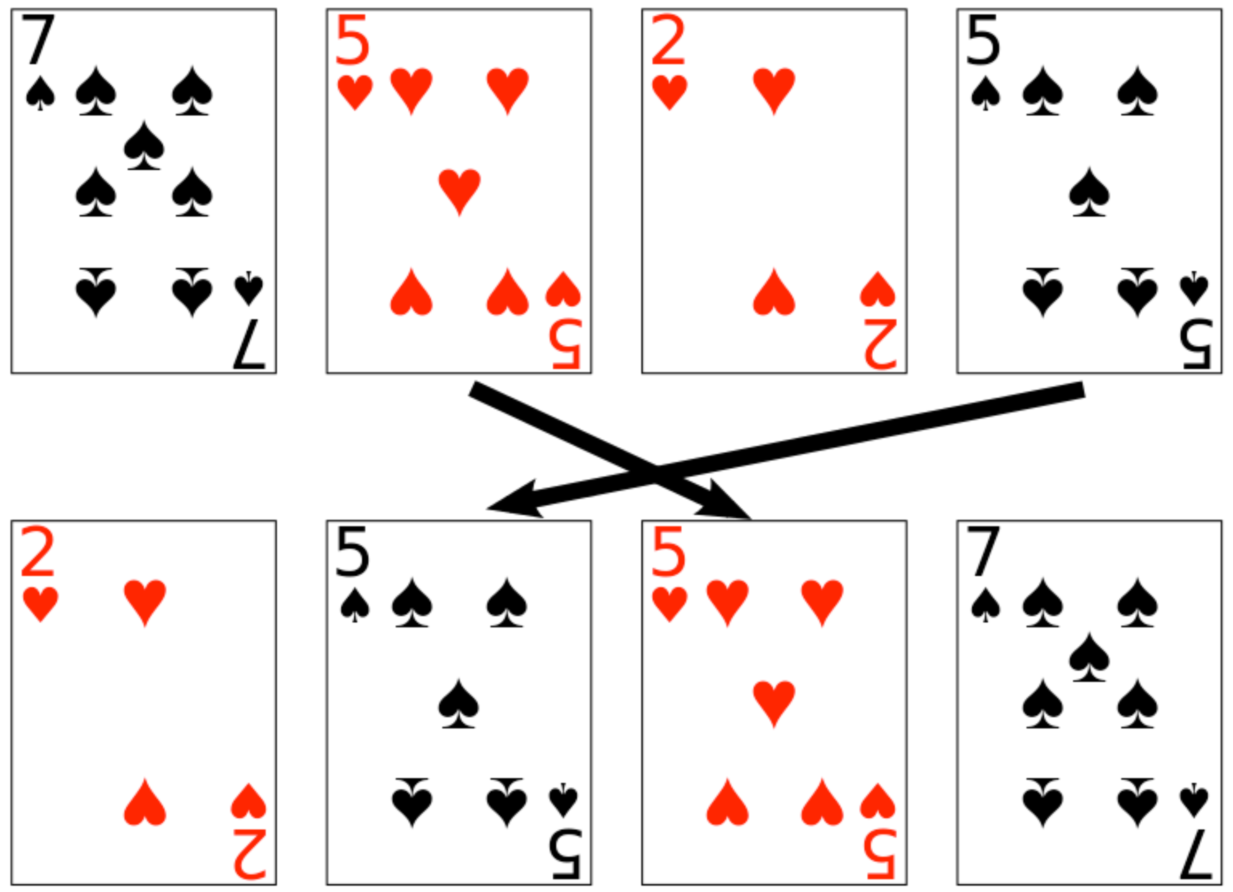
- In-place (原地): a sorting algorithm is in-place if $O(1)$ extra space is needed beyond input.

- Stability (稳定): a sorting algorithm is stable if numbers with the same value appear in the output array in the same order as they do in the input array.

Stable



Not stable



Sorting algorithms till now

Insertion Sort: gradually increase size of sorted part.

- $O(n^2)$ time, $O(1)$ space
- In-place, and stable.

Merge Sort: example of divide-and-conquer

- $O(n \log n)$ time, $O(n)$ space
- Not in-place, but stable.

Heap Sort: leverage the heap data structure

- $O(n \log n)$ time, $O(1)$ space
- In-place, but not stable.

Counterexample for stability: <2a, 2b, 1>:
It is already a max heap, then
1. 2a is extracted, and placed in the end
2. 2b is extracted, and placed in the end but one index

At last, we get <1, 2b, 2a>

Elementary sorting

The Selection Sort Algorithm

- **Basic idea**: pick out minimum element from input, then recursively sort remaining elements, and finally concatenate the minimum element with sorted remaining elements.

SelectionSortRec(A):

```
if |A| = 1
    return A
else
    min := GetMinElement(A)
    A' := RemoveElement(A, min)
    return Concatenate(min, SelectionSortRec(A'))
```

SelectionSort(A):

```
for i := 1 to A.length
    minIdx := i
    for j := i + 1 to A.length
        if A[j] < A[minIdx]
            minIdx := j
    Swap(i, minIdx)
```

Analysis of SelectionSort

- Why it is correct? (What is the loop invariant?)
  - After the i^{th} iteration, the first i items are sorted, and they are the i smallest elements in the original array.
- Time complexity for sorting n items?
  - $\sum_{i=1}^{n-1} (\Theta(1) + \Theta(n-i)) = \Theta(n^2)$

SelectionSort(A):

for $i := 1$ to $A.length$
  $minIdx := i$
  for $j := i + 1$ to $A.length$
    if $A[j] < A[minIdx]$
      $minIdx := j$
  $Swap(i, minIdx)$

Analysis of SelectionSort

- Space complexity?
  - $O(1)$ extra space, thus in-place
- Stability?
  - Not stable! Swap operation can mess up relative order
    - Counterexample for stability: <2a, 2b, 1>

SelectionSort(A):

for $i := 1$ to $A.length$
    $minIdx := i$
    for $j := i + 1$ to $A.length$
        if $A[j] < A[minIdx]$
            $minIdx := j$
    $Swap(i, minIdx)$

Before we move on

SelectionSort(A):

```
for  i := 1 to A.length
    minIdx := i
    for j := i + 1 to A.length
        if A[j] < A[minIdx]
            minIdx := j
    Swap(i, minIdx)
```

SelectionSortRec(A):

```
if  |A| = 1
    return A
else
    min := GetMinElement(A)
    A' := RemoveElement(A, min)
    return Concatenate(min, SelectionSortRec(A'))
```

Get the minimal element and extract it?

Similar operations: `HeapGetMax`, `HeapExtractMax`

Before we move on

SelectionSortRec(A):
if $|A| = 1$
  return A
else
  $min := GetMinElement(A)$
  $A' := RemoveElement(A, min)$
  return $Concatenate(min, SelectionSortRec(A'))$

SelectionSortRecVariant(A):
if $|A| = 1$
  return A
else
  $max := GetMaxElement(A)$
  $A' := RemoveElement(A, max)$
  return $Concatenate(SelectionSortRec(A'), max)$

Let A get organized as a heap, then it leads to the faster HeapSort algorithm.

The choice of data structure affects the performance of algorithms!

The Bubble Sort Algorithm

- **Basic idea**: repeatedly step through the array, compare adjacent pairs and swaps them if they are in the wrong order. Thus, larger elements "bubble" to the "top".

BubbleSort(A):

```
for  i := A.length down to 2
    for  j := 1 to i - 1
        if A[j] > A[j+1]
            Swap(A[j], A[j+1])
```

Analysis of BubbleSort

- Correctness:
  - What is the invariant?
- Time complexity:
  - $\Theta(n^2)$
- Space complexity:
  - $\Theta(1)$
- Stability:
  - Stable

BubbleSort(A):

for $i := A.length$ **down to** 2
    for $j := 1$ **to** $i - 1$
        if $A[j] > A[j+1]$
            Swap($A[j]$, $A[j+1]$)

Improving BubbleSort

```
BubbleSort(A):
for i := A.length down to 2
    for j := 1 to i - 1
        if A[j] > A[j+1]
            Swap(A[j], A[j+1])
```

- What if in one iteration we never swap data items?
  - Then $A[1 \ldots i]$ are sorted, and we are done! (Why?)

Improving BubbleSort

```
BubbleSortImporved(A):
n := A.length
repeat
    swapped := false
    for j := 1 to n - 1
        if A[j] > A[j+1]
            Swap(A[j], A[j+1])
            swapped := true
    n := n - 1
until swapped = false
```

- When the input is mostly sorted, this variant performs much better.
    - Particularly, when the input is sorted, this variant has $O(n)$ runtime.
        - Other algorithms that also have this property, E.g., InsertionSort.
    - Nonetheless, the worst case performance is still $\Theta(n^2)$.
        - E.g., when input is reversely sorted.

Improving BubbleSort

BubbleSortImporved(A):

```
n := A.length
repeat
    swapped := false
    for j := 1 to n - 1
        if A[j] > A[j+1]
            Swap(A[j], A[j+1])
            swapped := true
    n := n - 1
until swapped = false
```

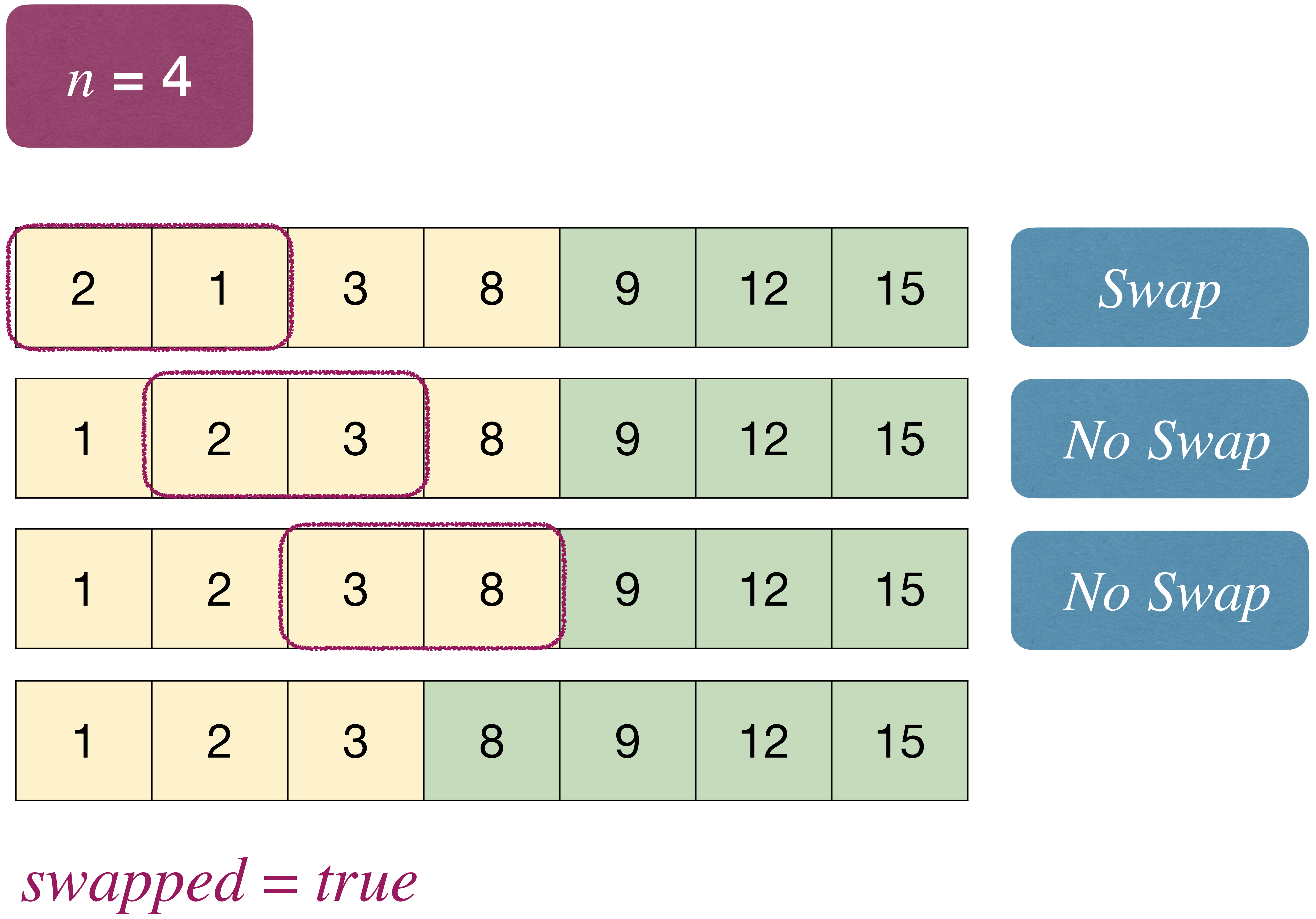
$n = 5$

3	2	1	8	9	12	15	*Swap*
2	3	1	8	9	12	15	*Swap*
2	1	3	8	9	12	15	*No Swap*
2	1	3	8	9	12	15	*No Swap*
2	1	3	8	9	12	15	

swapped = true

Improving BubbleSort

```
BubbleSortImporved(A):
n := A.length
repeat
    swapped := false
    for j := 1 to n - 1
        if A[j] > A[j+1]
            Swap(A[j], A[j+1])
            swapped := true
    n := n - 1
until swapped = false
```



Improving BubbleSort

BubbleSortImporved(A):

```
n := A.length
repeat
    swapped := false
    for j := 1 to n - 1
        if A[j] > A[j+1]
            Swap(A[j], A[j+1])
            swapped := true
    n := n - 1
until swapped = false
```

$n = 3$

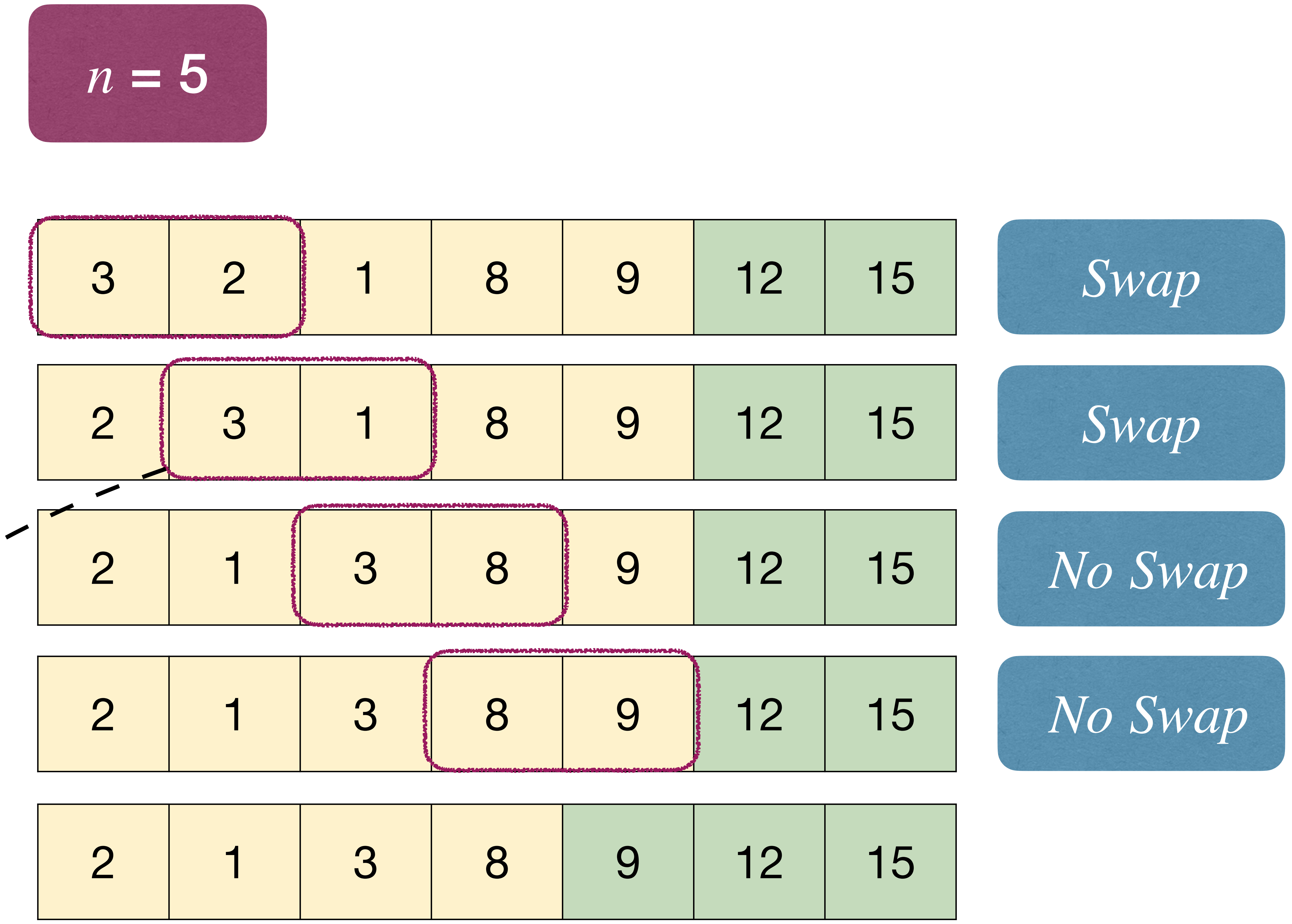
1	2	3	8	9	12	15	*No Swap*
1	2	3	8	9	12	15	*No Swap*
1	2	3	8	9	12	15	

swapped = false

Improving BubbleSort

BubbleSortImporved(A):

n := $A.length$
repeat
    $swapped$:= false
    for j := 1 **to** n - 1
        if $A[j] > A[j+1]$
            Swap($A[j]$, $A[j+1]$)
            $swapped$:= $true$
    n := n - 1
until $swapped$ = $false$



The last swap index is 2, and then the following items has no swap, indicating that the following items are already sorted!

Improving BubbleSort

BubbleSortImporved(A):

```
n := A.length
repeat
    swapped := false
    for j := 1 to n - 1
        if A[j] > A[j+1]
            Swap(A[j], A[j+1])
            swapped := true
    n := n - 1
until swapped = false
```

$n = 5$

3	2	1	8	9	12	15	*Swap*
2	3	1	8	9	12	15	*Swap*
2	1	3	8	9	12	15	*No Swap*
2	1	3	8	9	12	15	*No Swap*
2	1	3	8	9	12	15	

The last swap index is 2, and then the following items has no swap, indicating that the following items are already sorted!

Therefore, in the next step, n should be 2

Improving BubbleSort

- We can be more aggressive when reducing n after each iteration: in $A[1 \ldots n]$, items after the last swap are all in correct sorted position.

BubbleSortImporved(A):

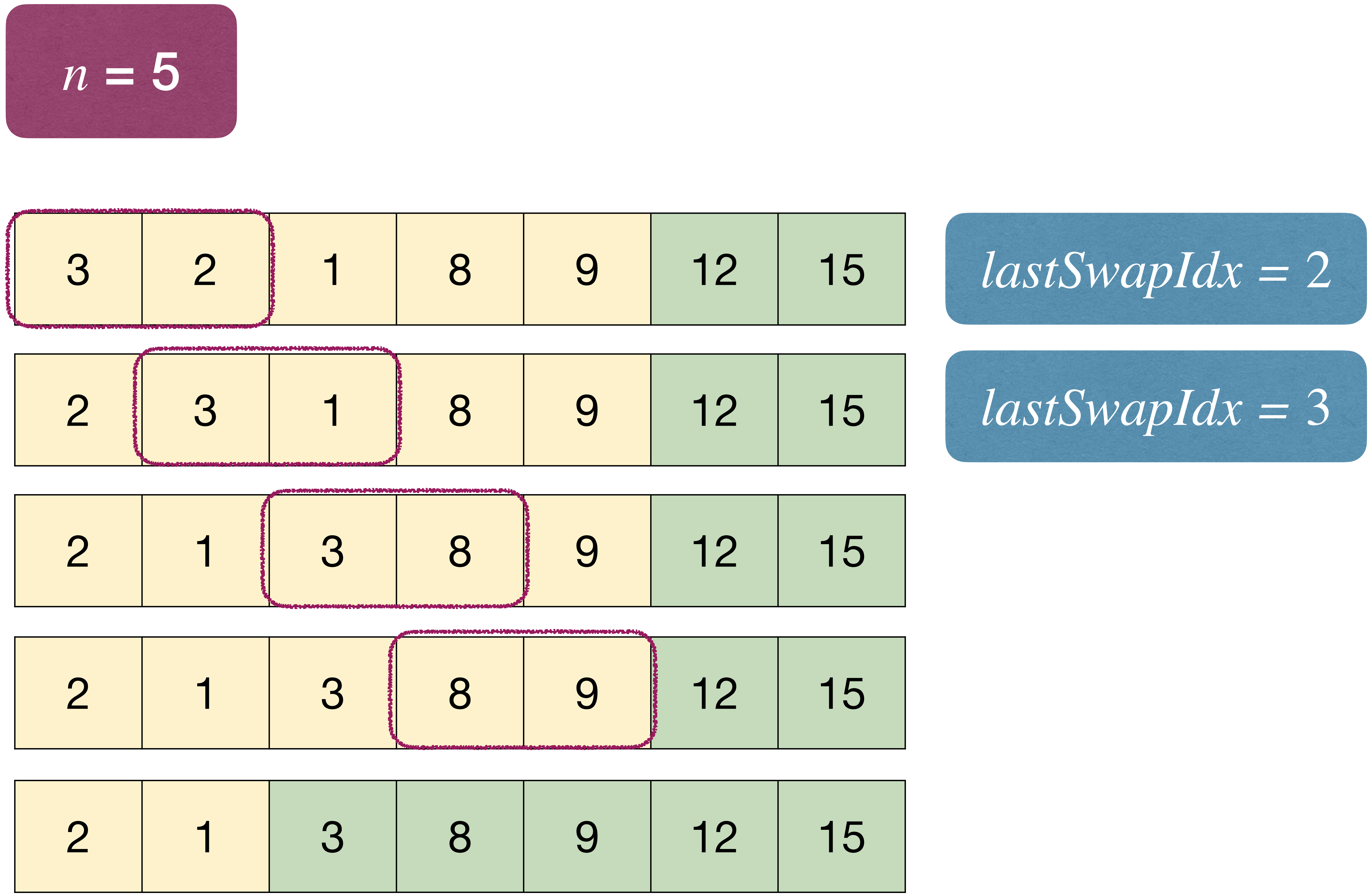
```
n := A.length
repeat
    swapped := false
    for j := 1 to n - 1
        if A[j] > A[j+1]
            Swap(A[j], A[j+1])
            swapped := true
    n := n - 1
until swapped = false
```

BubbleSortImporvedFurther(A):

```
n := A.length
repeat
    lastSwapIdx := -1
    for j := 1 to n - 1
        if A[j] > A[j+1]
            Swap(A[j], A[j+1])
            lastSwapIdx := j + 1
    n := lastSwapIdx - 1
until n <= 1
```

Improving BubbleSort

```
BubbleSortImporvedFurther(A):
n := A.length
repeat
    lastSwapIdx := -1
    for j := 1 to n - 1
        if A[j] > A[j+1]
            Swap(A[j], A[j+1])
            lastSwapIdx := j + 1
    n := lastSwapIdx - 1
until n <= 1
```



Improving BubbleSort

```
BubbleSortImporvedFurther(A):
n := A.length
repeat
    lastSwapIdx := -1
    for j := 1 to n - 1
        if A[j] > A[j+1]
            Swap(A[j], A[j+1])
            lastSwapIdx := j + 1
    n := lastSwapIdx - 1
until n <= 1
```

$n = 2$

2	1	3	8	9	12	15
1	2	3	8	9	12	15
1	2	3	8	9	12	15

$lastSwapIdx = 2$

$n = 1$ → break loop

Comparison of simple sorting algorithms

- Insertion
  - $n(n-1)/2$ **swaps,** and $n \cdot (n-1)/2$ **comparisons** -> worst
  - $n(n-1)/4$ **swaps,** and $n \cdot (n-1)/4$ **comparisons** -> on average
- Selection
  - $n-1$ **swaps,** and $n \cdot (n-1)/2$ **comparisons**
- Bubble
  - $n \cdot (n-1)/2$ **swaps,** and $n \cdot (n-1)/2$ **comparisons**

Recall the insertion sort....

Insertion-Sort(A):

```
for i := 2 to A.length
  key := A[i]
  j := i - 1
  while j > 0 and A[j] > key
      A[j + 1] := A[j]
      j := j - 1
  A[j + 1] := key
return A
```

Improving Insertion Sorting

- Insertion sorting is effective when:
  - Input size is small
  - The input array is nearly sorted (resulting in few comparisons and swaps)
- Insertion sorting is ineffective when:
  - Elements must move far in array

Improving Insertion Sorting

- Allow elements to move large steps
- Bring elements close to final location
  - Make array almost sorted

- Idea: for some decreasing step size h, e.g. $(\ldots, 8, 4, 2, 1)$, the sequence must end with 1 (to ensure the correctness of sorting)
  - For each step, sort the array so elements separated by exactly h elements apart are in order.

*Shell's method for sorting

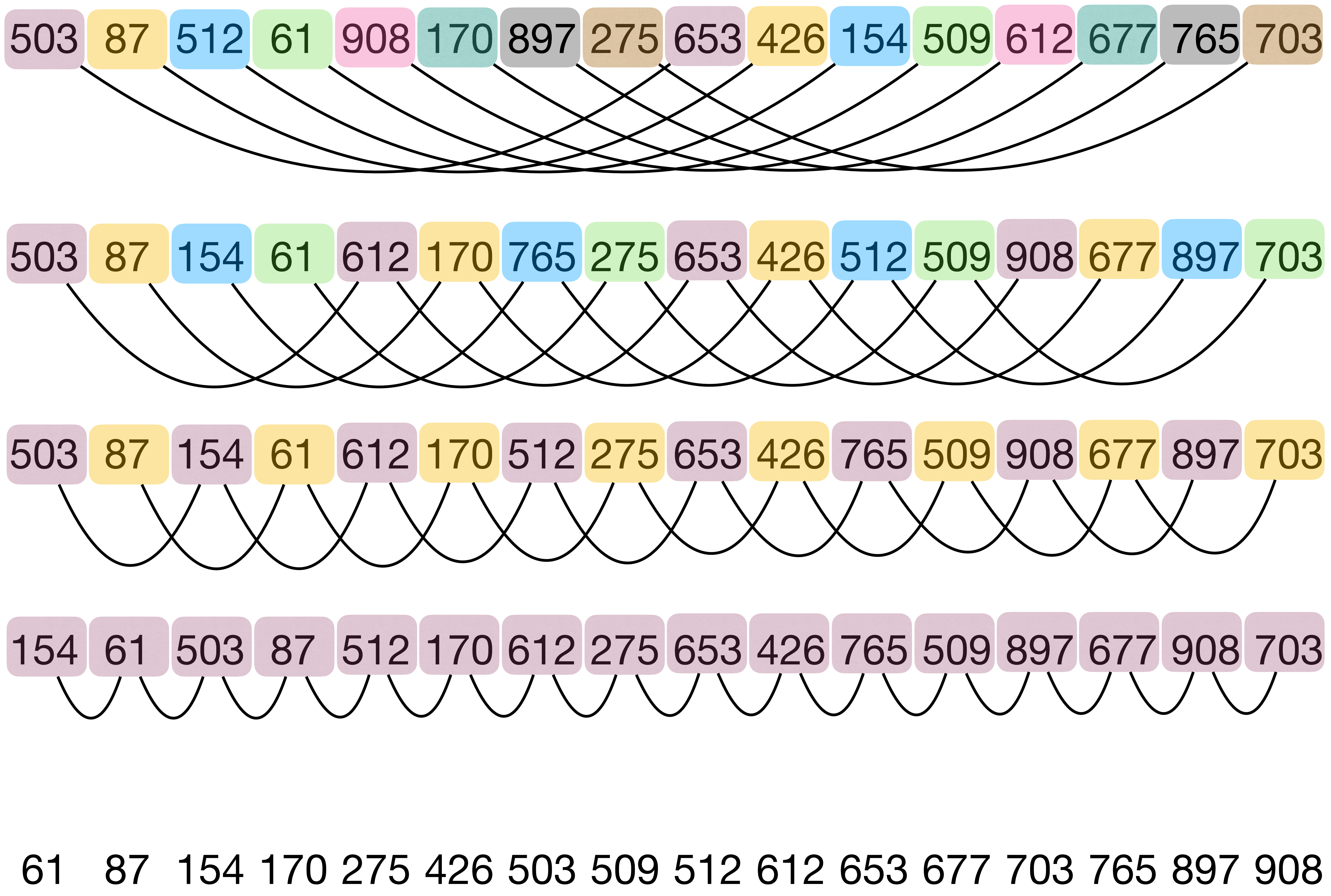
Let's first see an example of ShellSort: sort 16 integers.

[Pass 1] Group elements of distance **8** together, end up with eight groups each of size two. Sort these groups individually.

[Pass 2] Group elements of distance **4** together, end up with four groups each of size four. Sort these groups individually.

[Pass 3] Group elements of distance **2** together, end up with two groups each of size eight. Sort these groups individually.

[Pass 4] Group elements of distance **1**, this is just an ordinary sort on all elements.



General framework of ShellSort

- To sort n items, define a set of **decreasing distances** $\{d_1, d_2, \ldots, d_k\}$ with $d_1 < n$ and $d_k = 1$.
- `ShellSort` then go through k passes, for the i^{th} pass:
  - Divide items into d_i groups each of size about n/d_i, and the j^{th} group contains items with index $j, j + d_i, j + 2d_i, j + 3d_i, \cdots$
  - For each of the d_i groups, sort the items in that group. (uses `InsertionSort`.)

Donald L. Shell

Benefit of ShellSort

- In a sequence of items $\langle a_1, a_2, \cdots, a_n \rangle$, if $i < j$ and $a_i > a_j$, then the pair (a_i, a_j) is call an **inversion**.
- The process of sorting is to correct all inversions!
- Earlier passes in `ShellSort` reduce number of inversions, making the sequence "closer" to being sorted.
- `InsertionSort` performs better (i.e., faster) as the input sequence becomes "closer" to being sorted.

Ideal versus Reality

- Unfortunately, `ShellSort` is not that fast, at least when using Shell's original distances…
- Upper bound on the runtime of `ShellSort`:
  - Assume we have n items where n is some power of two.
  - The distances are $n/2, n/4, \ldots, 1$.
  - For the i^{th} pass, we run $n/2^i$ instances of `InsertionSort`, each having to sort 2^i items.
  - So the total runtime is $\sum_{i=1}^{(\lg n)-1} (n/2^i \cdot O(2^{2i})) = O(n^2)$
- Will `ShellSort` actually perform so poor?

ShellSort can be slow!

- When using Shell's original distances, the runtime of `ShellSort` can be $\Theta(n^2)$ for certain input sequences.

- Example: input is $[n]$, where $[n/2]$ are in even positions, and $[n]\backslash[n/2]$ are in odd positions.

8	0	9	1	10	2	11	3	12	4	13	5	14	6	15	7

- Then, before the last pass, no pair (a_i, a_j) where i and j are of different parity is ever compared!

- In the last pass, $\Theta(n^2)$ work has to be done!

Choice of distances matters, a lot!

General term ($k \geq 1$)	Concrete gaps	Worst-case time complexity	Author and year of publication
$\left\lfloor \frac{N}{2^k} \right\rfloor$	$1, 2, \ldots, \left\lfloor \frac{N}{4} \right\rfloor, \left\lfloor \frac{N}{2} \right\rfloor$	$\Theta\left(N^2\right)$ [e.g. when $N = 2^p$]	Shell, 1959[4]
$2\left\lfloor \frac{N}{2^{k+1}} \right\rfloor + 1$	$1, 3, \ldots, 2\left\lfloor \frac{N}{8} \right\rfloor + 1, \; 2\left\lfloor \frac{N}{4} \right\rfloor + 1$	$\Theta\left(N^{\frac{3}{2}}\right)$	Frank & Lazarus, 1960[8]
$2^k - 1$	$1, 3, 7, 15, 31, 63, \ldots$	$\Theta\left(N^{\frac{3}{2}}\right)$	Hibbard, 1963[9]
$2^k + 1$, prefixed with 1	$1, 3, 5, 9, 17, 33, 65, \ldots$	$\Theta\left(N^{\frac{3}{2}}\right)$	Papernov & Stasevich, 1965[10]
Successive numbers of the form $2^p 3^q$ (3-smooth numbers)	$1, 2, 3, 4, 6, 8, 9, 12, \ldots$	$\Theta\left(N \log^2 N\right)$	Pratt, 1971[1]
$\frac{3^k - 1}{2}$, not greater than $\left\lceil \frac{N}{3} \right\rceil$	$1, 4, 13, 40, 121, \ldots$	$\Theta\left(N^{\frac{3}{2}}\right)$	Knuth, 1973,[3] based on Pratt, 1971[1]
$\prod_I a_q$, where $a_0 = 3$, $a_q = \min\left\{n \in \mathbb{N} \colon n \geq \left(\frac{5}{2}\right)^{q+1}, \forall p \colon 0 \leq p < q \Rightarrow \gcd(a_p, n) = 1\right\}$, $I = \left\{0 \leq q < r \mid q \neq \frac{1}{2}\left(r^2 + r\right) - k\right\}$, $r = \left\lfloor \sqrt{2k + \sqrt{2k}} \right\rfloor$	$1, 3, 7, 21, 48, 112, \ldots$	$O\left(N^{1+\sqrt{\frac{8\ln(5/2)}{\ln(N)}}}\right)$	Incerpi & Sedgewick, 1985,[11] Knuth[3]

Quick Sort

A unified view of many sorting algorithms

Divide problem into subproblems. **Conquer** subproblems recursively. **Combine** solutions of subproblems.

- Divide the input into size 1 and size n - 1.
  - `InsertionSort`, easy to divide, combine needs efforts.
  - `SelectionSort`, divide needs efforts, easy to combine.
- Divide the input into two parts each of same size.
  - `MergeSort`, easy to divide, combine needs efforts.
- Divide the input into two parts of **approximately** same size.
  - `QuickSort`, divide needs efforts, easy to combine.

The QuickSort Algorithm

- **Basic idea**:
  - Given an array A of n items.
    - Choose one item x in A as the **pivot**.
    - Use the pivot to **partition** the input into B and C, so that items in B are $\leq x$, and items in C are $> x$.
    - Recursively sort B and C.
    - Output $\langle B, x, C \rangle$.

QuickSortAbs(A):

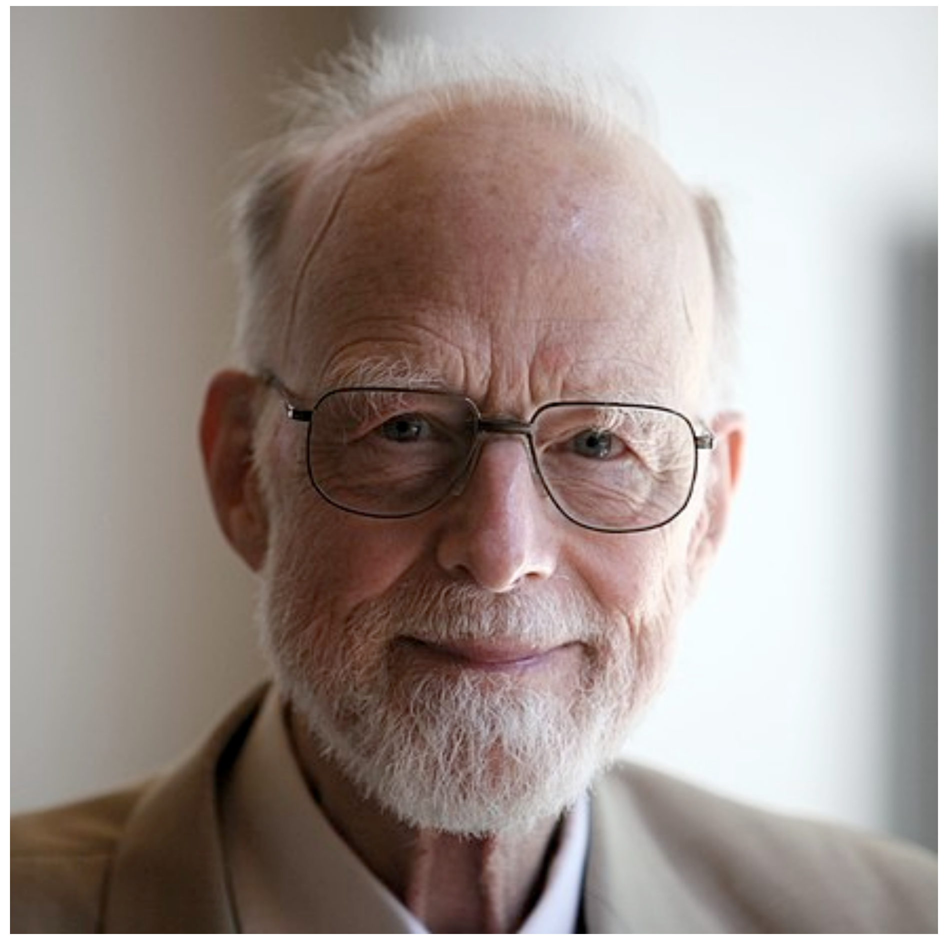
$x := GetPivot(A)$

$<B, C> := Partition(A, x)$

$QuickSortAbs(B)$

$QuickSortAbs(C)$

return $Concatenate(B, x, C)$



Tony Hoare

Choosing the pivot

- Ideally the pivot should partition the input into two parts of roughly the same size (we'll see why later).
  - Select the "middle" element, the "first" element, or the "last" element?
  - Or using "Median-of-three" technique, e.g., $A[1], A[n], A[n/2]$, median of $\{A[1], A[n], A[n/2]\}$?
- For every **simple deterministic** method of choosing pivot, we can construct corresponding "**bad input**".
- For now just use the **last** item as the pivot.

The Partition Procedure

- Allocate array B of size n.
- Sequentially go through $A[1\ldots(n\text{-}1)]$, put small items at the left side of B, and large items at the right side of B.
- Finally put the pivot in the (only) remaining position.
- $\Theta(n)$ time, $\Theta(n)$ space, unstable.
- Can we do better, and how?

Partition(A):

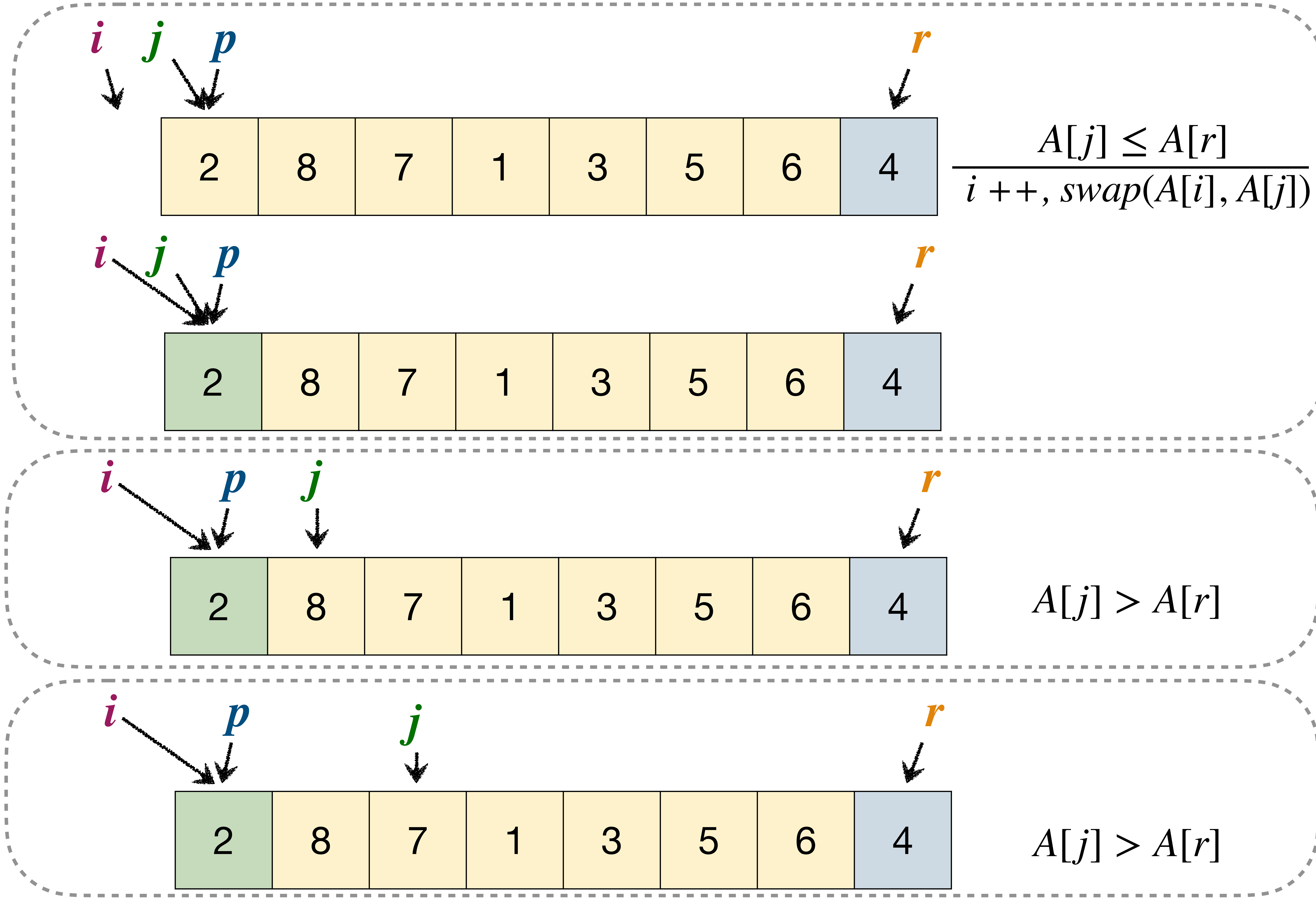
```
x := A[n],  l := 1, r := n
for  i := 1 to n - 1
    if  A[i] <= x
        B[l] := A[i]
        l++
    else
        B[r] := A[i]
        r--
B[l] := x
return < B, l >
```

In-place Partition Procedure

- Basic idea: sequentially go through A, use swap operations to move small items to the left part of A; thus the right part of A naturally contains large items.

InplacePartition(A, p, r):

```
i := p - 1
for j := p to r - 1
    if A[j] <= A[r]
        i := i + 1
        Swap(A[i], A[j])
Swap(A[i+1], A[r])
return i + 1
```

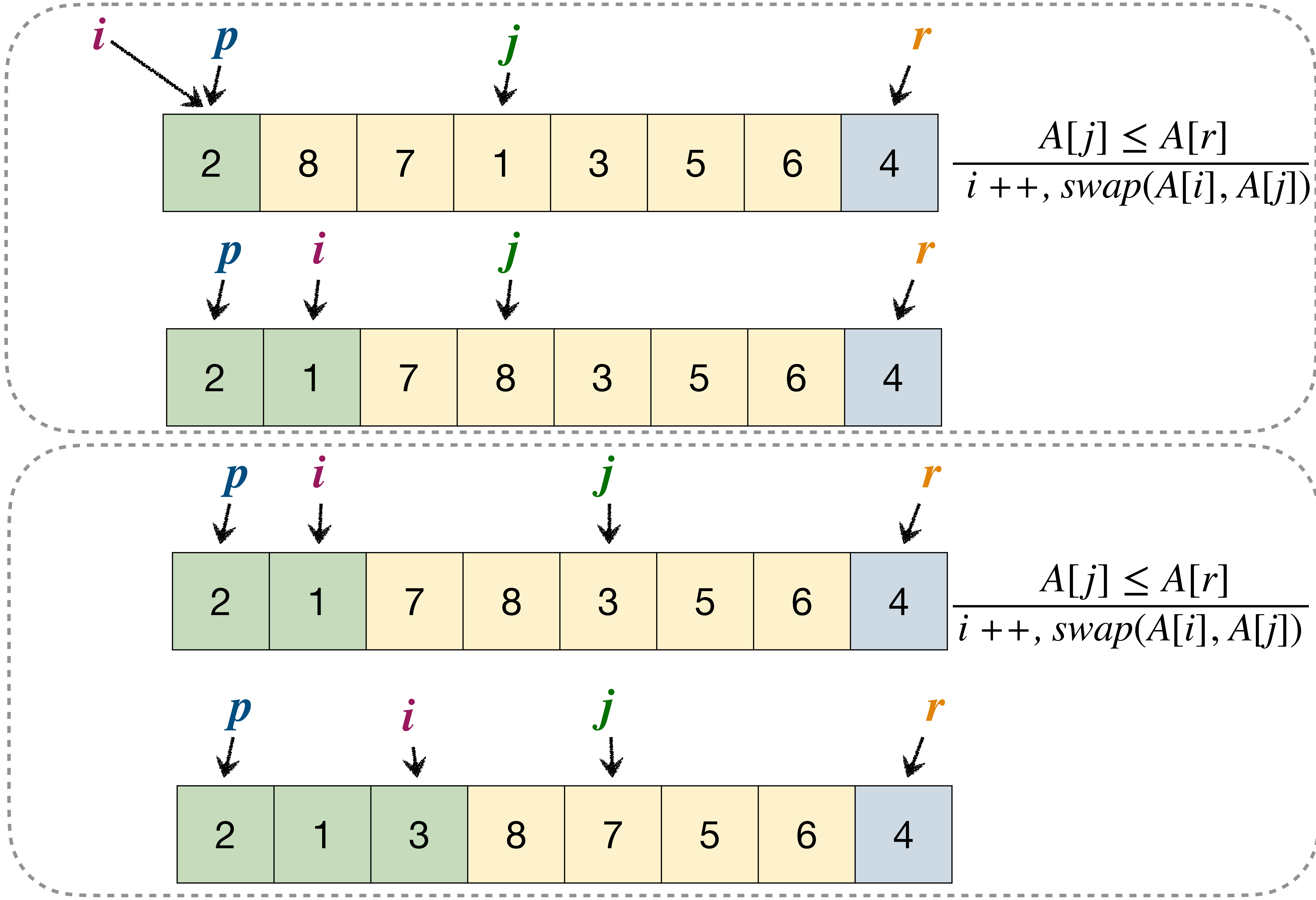


In-place Partition Procedure

- Basic idea: sequentially go through A, use swap operations to move small items to the left part of A; thus the right part of A naturally contains large items.

InplacePartition(A, p, r):

$i := p - 1$
for $j := p$ **to** $r - 1$
  if $A[j] <= A[r]$
    $i := i + 1$
    $Swap(A[i], A[j])$
$Swap(A[i+1], A[r])$
return $i + 1$

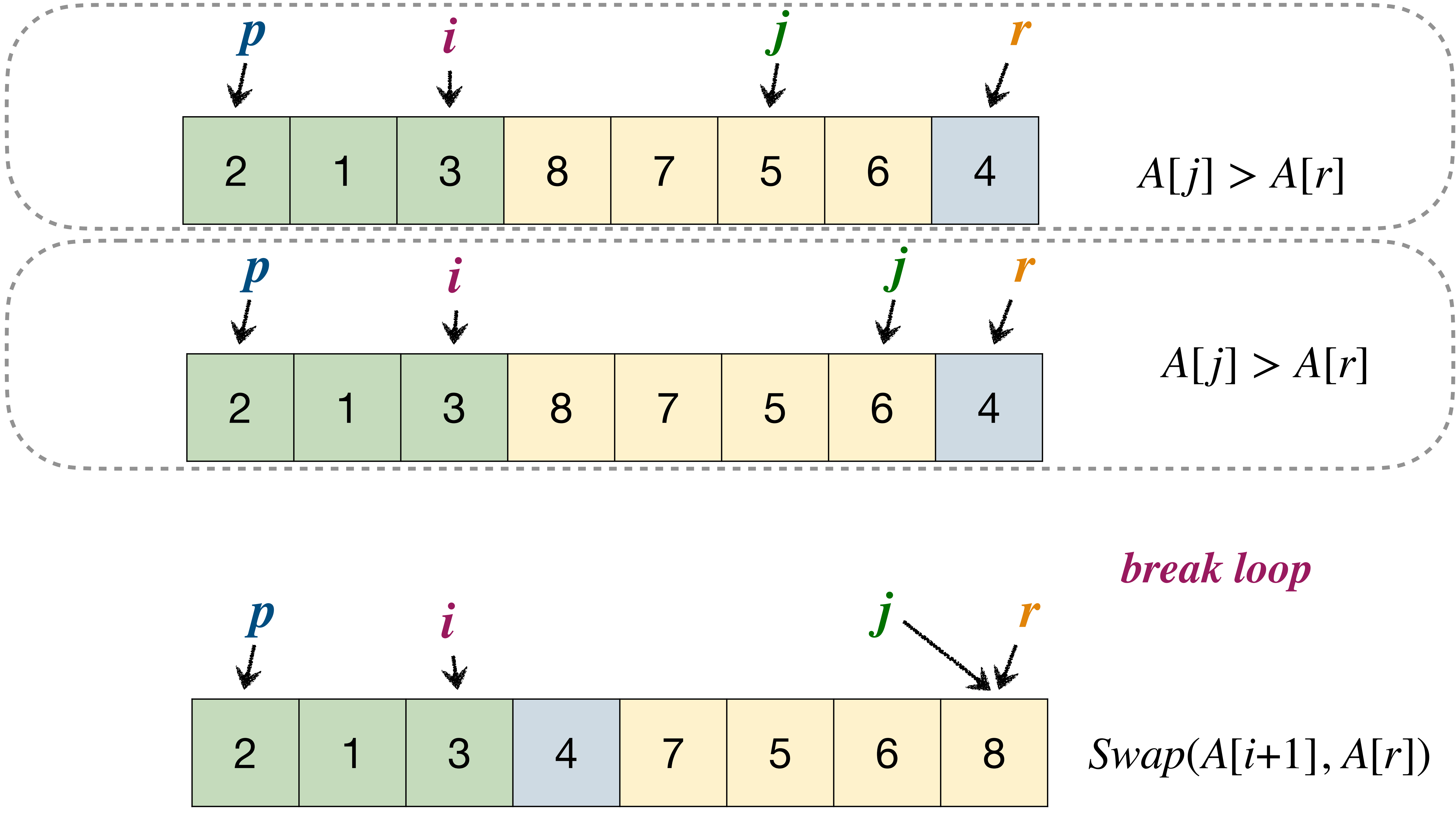


In-place Partition Procedure

- Basic idea: sequentially go through A, use swap operations to move small items to the left part of A; thus the right part of A naturally contains large items.

<u>InplacePartition(A, p, r):</u>

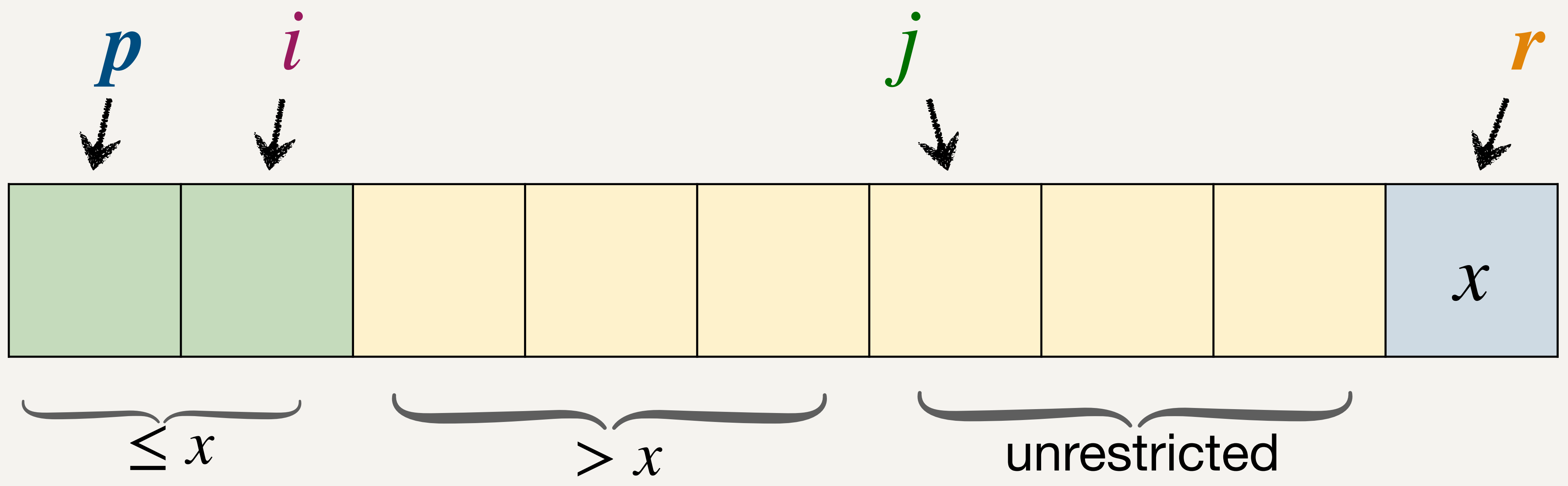
$i := p - 1$
for $j := p$ **to** $r - 1$
  if $A[j] <= A[r]$
    $i := i + 1$
    $Swap(A[i], A[j])$
$Swap(A[i+1], A[r])$
return $i + 1$



Analysis of In-place Partition Procedure

Correctness

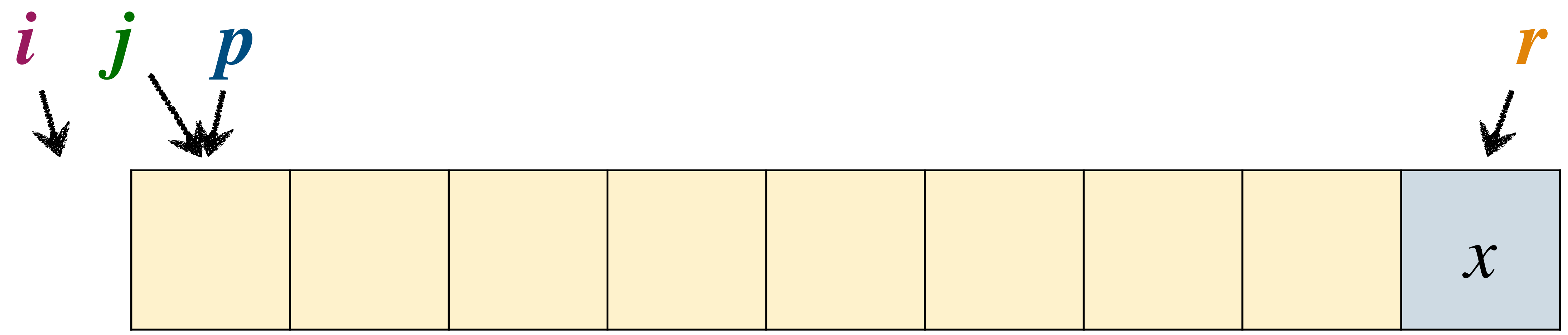
- **Claim**: at the beginning of any iteration, for any index k:
  - If $k \in [p, i]$, then $A[k] \leq A[r]$;
  - If $k \in [i+1, j-1]$, then $A[k] > x$;
  - If $k = r$, then $A[k] = A[r]$.



InplacePartition(A, p, r):

```
i := p - 1
for j := p to r - 1
    if A[j] <= A[r]
        i := i + 1
        Swap(A[i], A[j])
Swap(A[i+1], A[r])
return i + 1
```

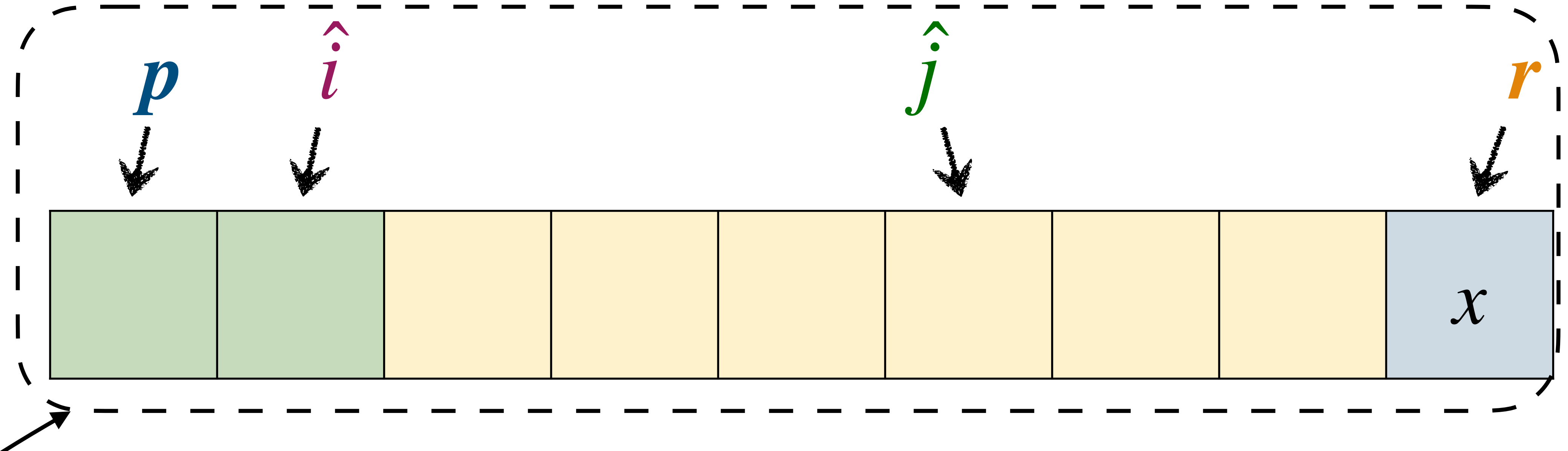
- **Proof**: we use induction.
  - **[Basis]** Trivially holds.



Analysis of In-place Partition Procedure

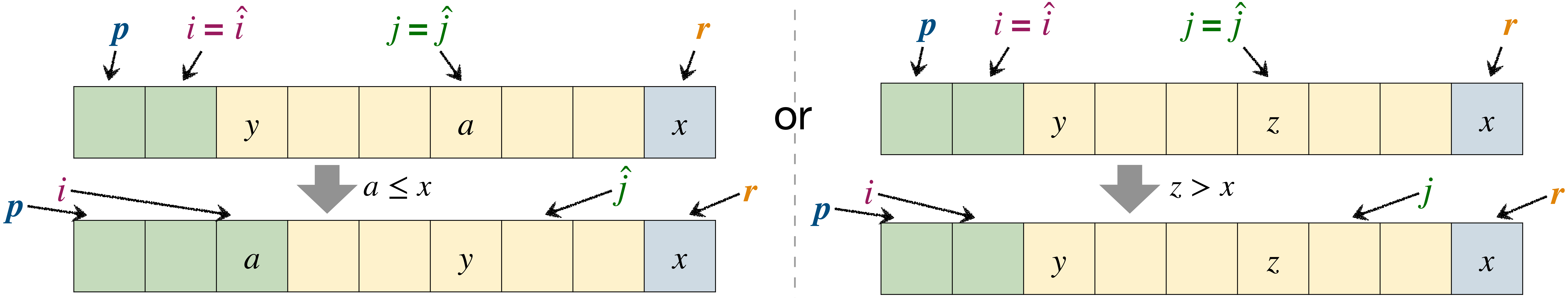
Correctness

- **Proof**: we use induction.
  - **[Basis]** Trivially holds.
  - **[Inductive step]** Assume at the beginning of some iteration we have $i = \hat{i}$ and $j = \hat{j}$, and the stated properties hold. In this iteration:



InplacePartition(A, p, r):

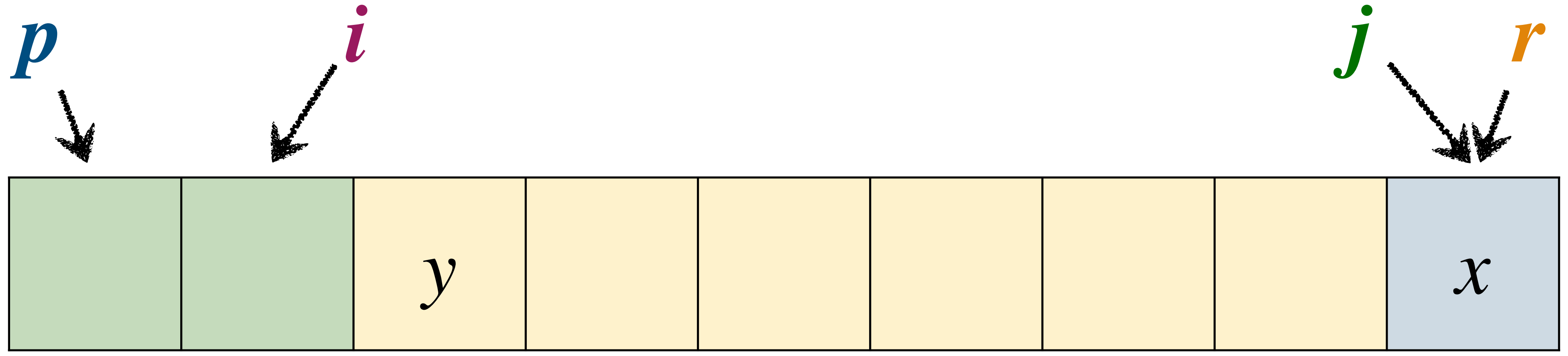
```
i := p - 1
for j := p to r - 1
    if A[j] <= A[r]
        i := i + 1
        Swap(A[i], A[j])
Swap(A[i+1], A[r])
return i + 1
```



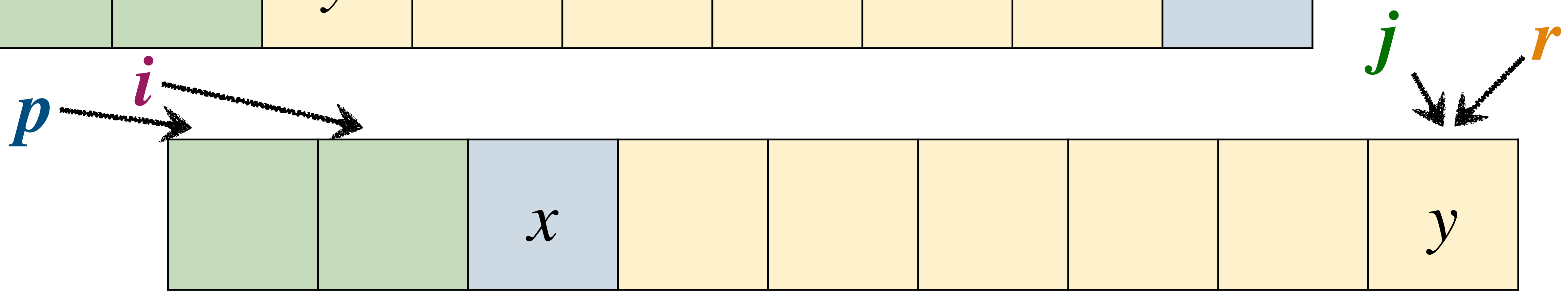
Analysis of In-place Partition Procedure

Correctness

- **Proof**: we use induction.
  - **[Basis]** Trivially holds.
  - **[Inductive step]** Assume at the beginning of some iteration we have $i = \hat{i}$ and $j = \hat{j}$, and the stated properties hold. Then they hold after this iteration.
    - eventually, when $j = r$:



    - Swap $A[i+1]$ and $A[r]$




```
InplacePartition(A, p, r):
i := p - 1
for  j := p to r - 1
    if  A[j] <= A[r]
        i := i + 1
        Swap(A[i], A[j])
Swap(A[i+1], A[r])
return i + 1
```

During execution, we only *swap* items, no *addition /deletion*.
So *InplacePartition* correctly partitions the input array.

The QuickSort Algorithm

```
InplacePartition(A, p, r):
i := p - 1
for j := p to r - 1
    if A[j] <= A[r]
        i := i + 1
        Swap(A[i], A[j])
Swap(A[i+1], A[r])
return i + 1
```

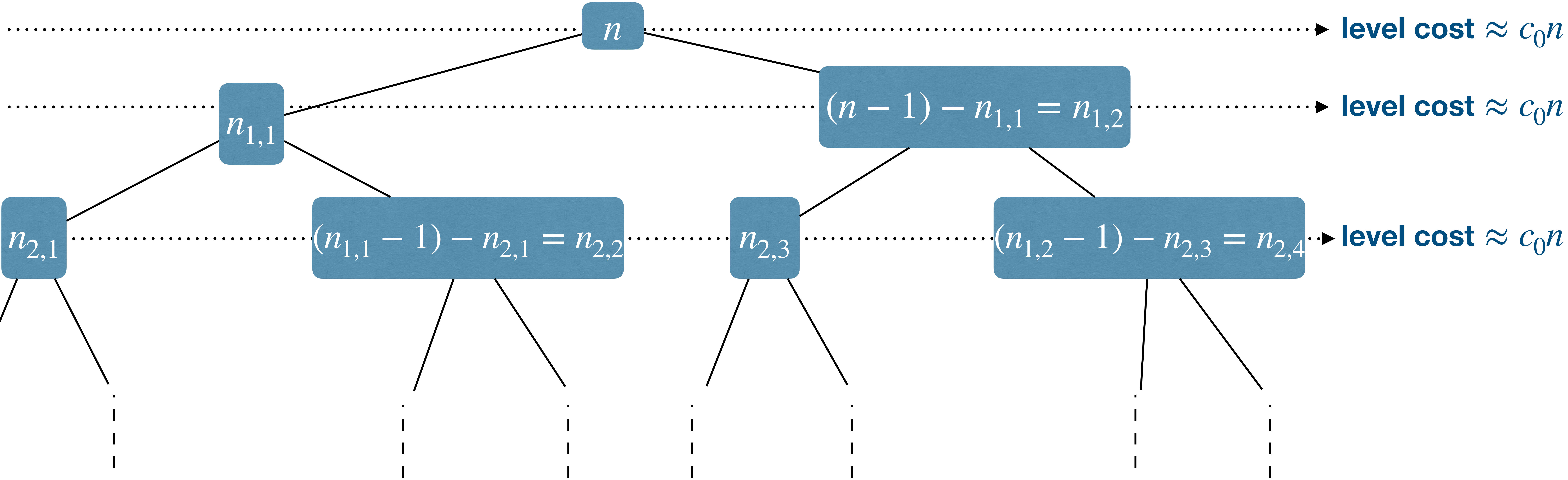
```
QuickSort(A, p, r):
if p < r
    q := InplacePartition(A, p, r)
    QuickSort(A, p, q - 1)
    QuickSort(A, q + 1, r)
```

- Performance of `InplacePartition`:
  - $\Theta(|r-p|)$ time (i.e., linear time);
  - $O(1)$ space; unstable.
- Performance of `QuickSort`?

Note: Although quicksort sorts in-place, the amount of memory it use aside from the array being sorted is not constant.
Since each recursive call requires additional amount of space on the runtime stack. How many of them?

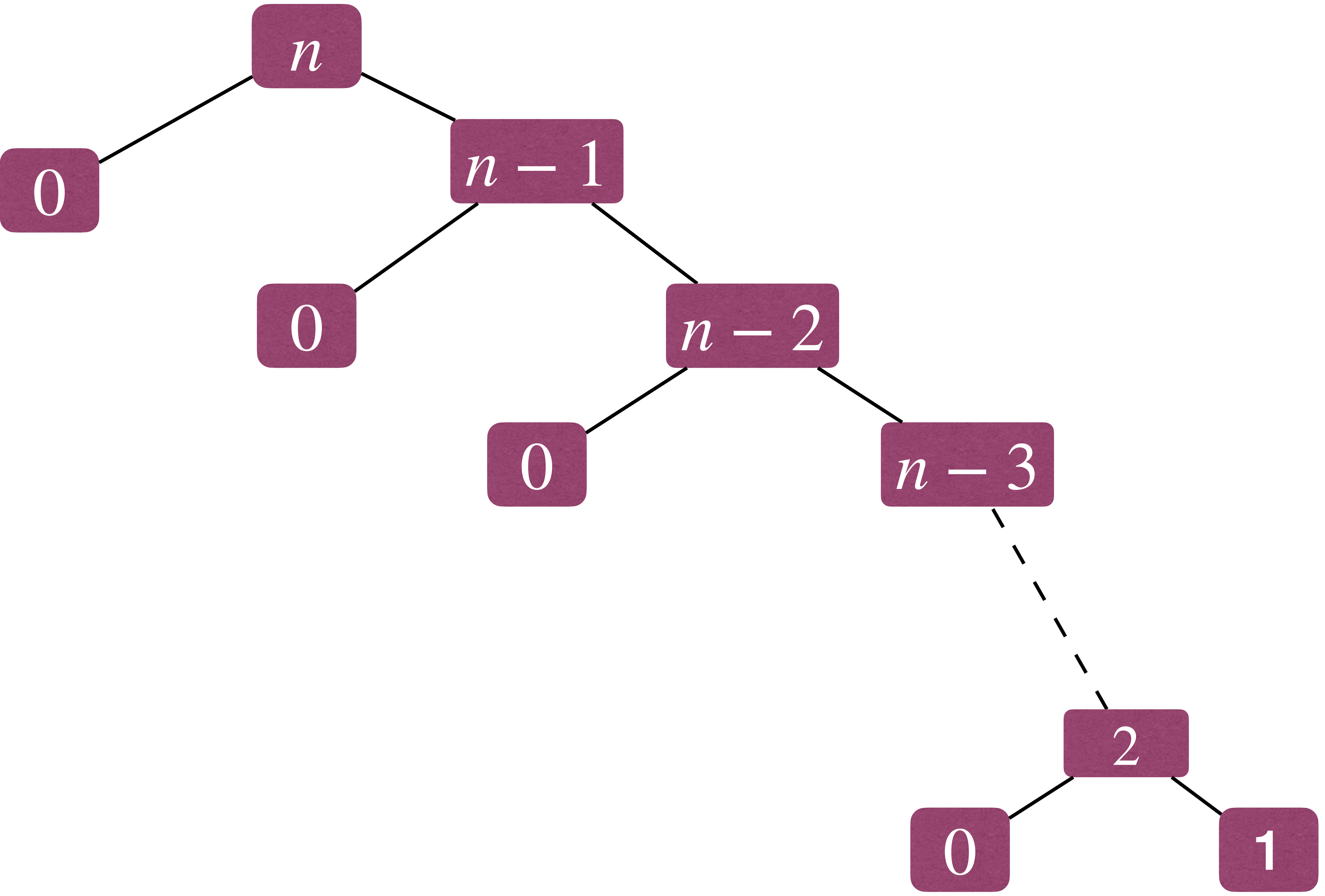
Performance of QuickSort

- Cost at each level is: $c_0(n-m)$, where m is number of pivots removed in lower level `Partition`.
    - If the partition is "balanced", then there will be few levels.
    - If the partition is "balanced", then m will increase rapidly.

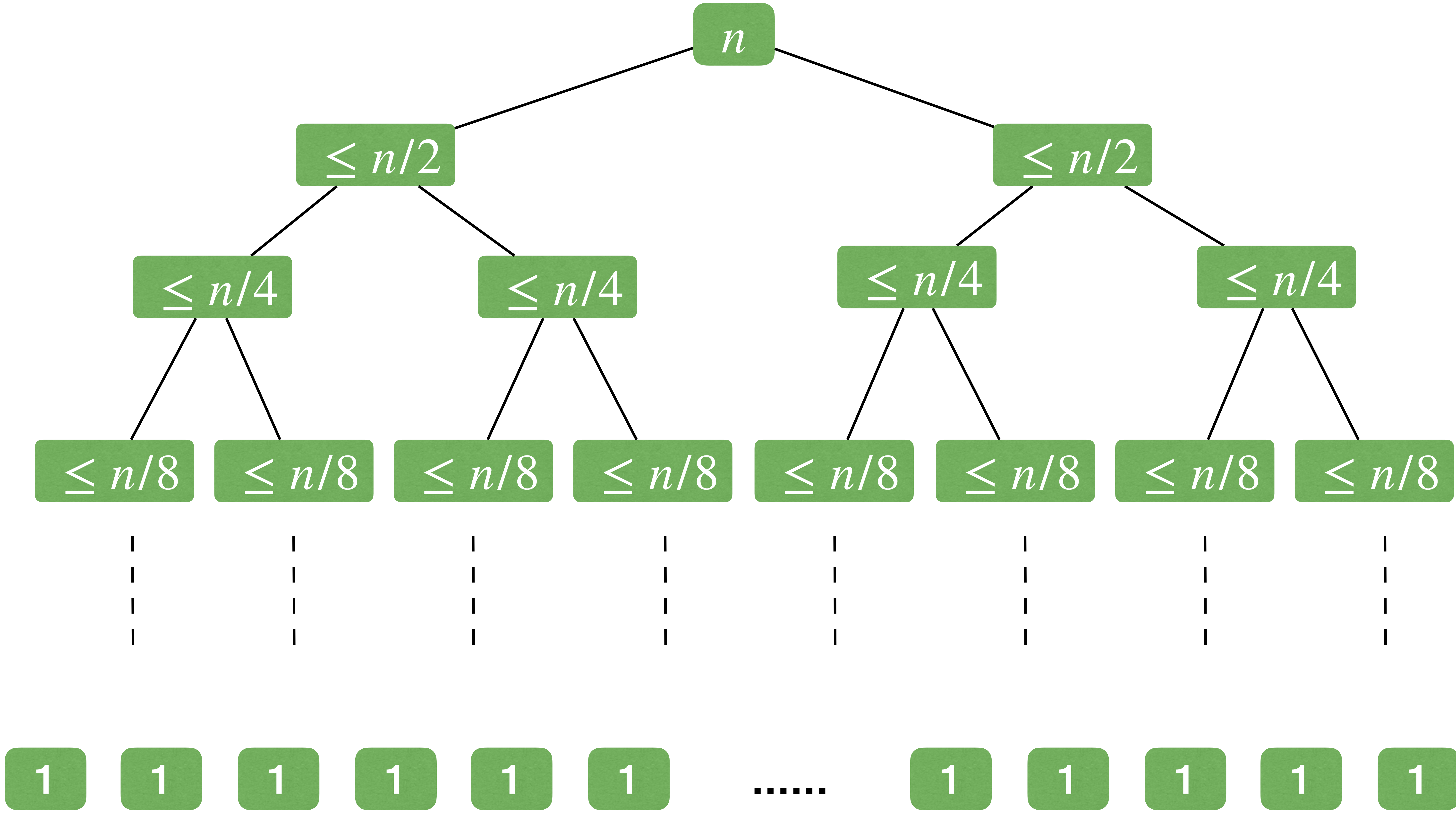


Performance of QuickSort

Worst case



Best case



Performance of QuickSort

- Recurrence for the worse-case runtime of `QuickSort`:
  - $T(n) = \max_{0 \le q \le n-1} (T(q) + T(n-q-1)) + c_0 n$
- Guess $T_n \le cn^2$, and we now verify:
  - $T(n) \le \max_{0 \le q \le n-1} (cq^2 + c(n-q-1)^2) + c_0 n$

$$= c \cdot \max_{0 \le q \le n-1} (q^2 + (n-q-1)^2) + c_0 n$$

when $q = 0$ or $q = n - 1$

$$\le c(n-1)^2 + c_0 n = cn^2 - c(2n-1) + c_0 n$$

$$\le cn^2 \qquad \rightarrow T(n) = O(n^2)$$

QuickSort(A, p, r):

if $p < r$

$q := InplacePartition(A, p, r)$

$QuickSort(A, p, q - 1)$

$QuickSort(A, q + 1, r)$

Performance of QuickSort

- "Balanced" partition gives best case performance.
  - $T(n) \leq T(n/2) + T(n/2) + \Theta(n)$ implies $T(n) = O(n \log n)$.

QuickSort(A, p, r):
if $p < r$
$q := InplacePartition(A, p, r)$
$QuickSort(A, p, q - 1)$
$QuickSort(A, q + 1, r)$

- `Partition` does not need to be perfectly balanced, we only need each split to be **constant** proportionality.
  - $T(n) \leq T(dn) + T((1-d)n) + \Theta(n)$ where $d = \Theta(1)$.

Performance of QuickSort

Performance of QuickSort

- The performance of the best is $\Theta(n \log n)$, while the worst is $\Theta(n^2)$
  - What about the performance in general?

- **Average-case analysis:** the **expected** time of algorithm over all inputs of size n (i.e., $\mathscr{X}_n$) : $A(n) = \sum_{x \in \mathscr{X}_n} T(x) \cdot Pr(x)$
  - In order to perform a probabilistic analysis, we must use knowledge of, or make assumptions about, the distribution of (something about) the inputs.

Performance of QuickSort

- For `QuickSort`, particular values in the array are not important, instead, the **relative ordering** of the values is what matters (since `QuickSort` is comparison-based).

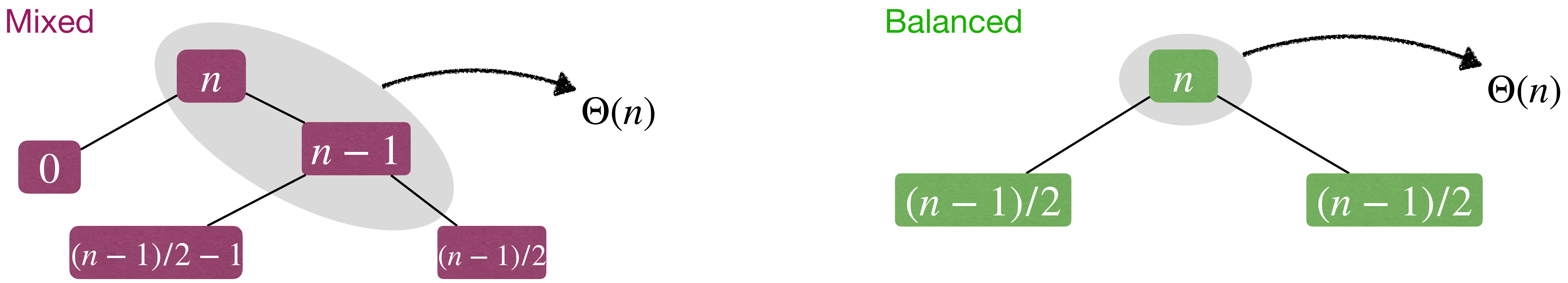
- Therefore, it is important to focus on the permutation of input numbers. A readable assumption is that all permutations of the input numbers are equally likely.

  - To make the analysis simple, we also assume that the elements are distinct (duplicate values will be discussed later).

Performance of QuickSort

- Before making rigorous analysis, we can first gain some intuition about the average performance.
  - When `QuickSort` runs on a random input array, we expect that some of the splits will be reasonably **well balanced** and that some will be **fairly unbalanced.**
  - In the average case, `Partition` produces a **mix** of "**good**" and "**bad**" splits. That is, in a recursion tree, the good and bad splits are distributed randomly throughout the tree.

Performance of QuickSort



- Further, for the sake of intuition, suppose that the good and bad splits **alternate** levels in the tree, and that the good splits are best case splits and the bad splits are worst-case splits.
  - As an example in the above, the "mixed" `Partition` produces two "(n-1)/2" subarrays at the cost of $\Theta(n) + \Theta(n-1) = \Theta(n)$, while the "balanced" `Partition` does so at the cost of $\Theta(n)$.
  - The cost of "bad" `Partition` can be absorbed by recent "good" `Partition`, without affecting time complexity asymptotically —> "mixed" `Partition` is at most constant factor worse than "balanced" `Partition`.
  - Therefore, the **average runtime** of QuickSort is $O(n \log n)$ (rigorously proved later).

Randomized QuickSort

- Picking "good" pivot is important for the performance? but how do we do it?
  - On choosing pivot: first, middle, last, median of three, …?
- Any simple **deterministic** mechanism could fail! (If the input is given by an "adversary" that knows the algorithm.)

- Choose pivot (uniformly) **at random**!
  - Since the choice is randomly made, there is a good chance (constant probability) that we choose a "good" pivot.
- The above claim holds even if the input is given by an "adversary" that knows the algorithm (but not the random bits the algorithm uses).

RandQuickSort(A, p, r):

if $p < r$

$i := Random(p, r)$

$Swap\ (A[r], A[i])$

$q := InplacePartition(A, p, r)$

$RandQuickSort(A, p, q - 1)$

$RandQuickSort(A, q + 1, r)$

Randomized QuickSort

RandQuickSort(A, p, r):
if $p < r$
  $i := Random(p, r)$
  $Swap(A[r], A[i])$
  $q := InplacePartition(A, p, r)$
  $RandQuickSort(A, p, q - 1)$
  $RandQuickSort(A, q + 1, r)$

Constant time

Two calls

InplacePartition(A, p, r):
$i := p - 1$
for $j := p$ **to** $r - 1$
  if $A[j] <= A[r]$
    $i := i + 1$
    $Swap(A[i], A[j])$
$Swap(A[i+1], A[r])$
return $i + 1$

O(**number of comparsions**)

Randomized QuickSort

- Cost of a call to `RndQuickSort`:
  - Choose a pivot in $\Theta(1)$ time;
  - Run `InplacePartition`, the cost is O(number of comparsions).
  - Need to call `RndQuickSort` twice, the calling process (**not** the subroutines themselves) needs $\Theta(1)$ time.

- Total cost of `RndQuickSort`:
  - Time for choosing pivots $O(n)$, since each node can be pivot at most once!
  - All calls to `InplacePartition`, O(total number of comparions).
  - Total time for call `RndQuickSort` $O(2n)$, since each time a pivot is chosen, two `RndQuickSort` calls are made.

In an execution of `RndQuickSort`, the cost is $O(n)$ + O(total number of comparions)

Randomized QuickSort

Cost of `RndQuickSort` is $O(n+X)$, where X is a random variable denoting the number of comparisons happened in `InplacePartition` throughout entire execution.

- Each of pair of items is compared at most once! (Items only compare with pivots, and each item can be the pivot at most once.)

- For ease of analysis, we let's index the elements of the array A by their position in the **sorted** output, rather than their **position** in the input.
  - For all the elements, we refer them to be $z_1, z_2, \ldots z_n$, with $z_1 < z_2 < \ldots < z_n$.

- Let $X_{ij} = I\{z_i \text{ is ever compared to } z_j\}$, here I is an indicator random variable $I(H) = \begin{cases} 1 & H \text{ happens} \\ 0 & H \text{ not happen} \end{cases}$

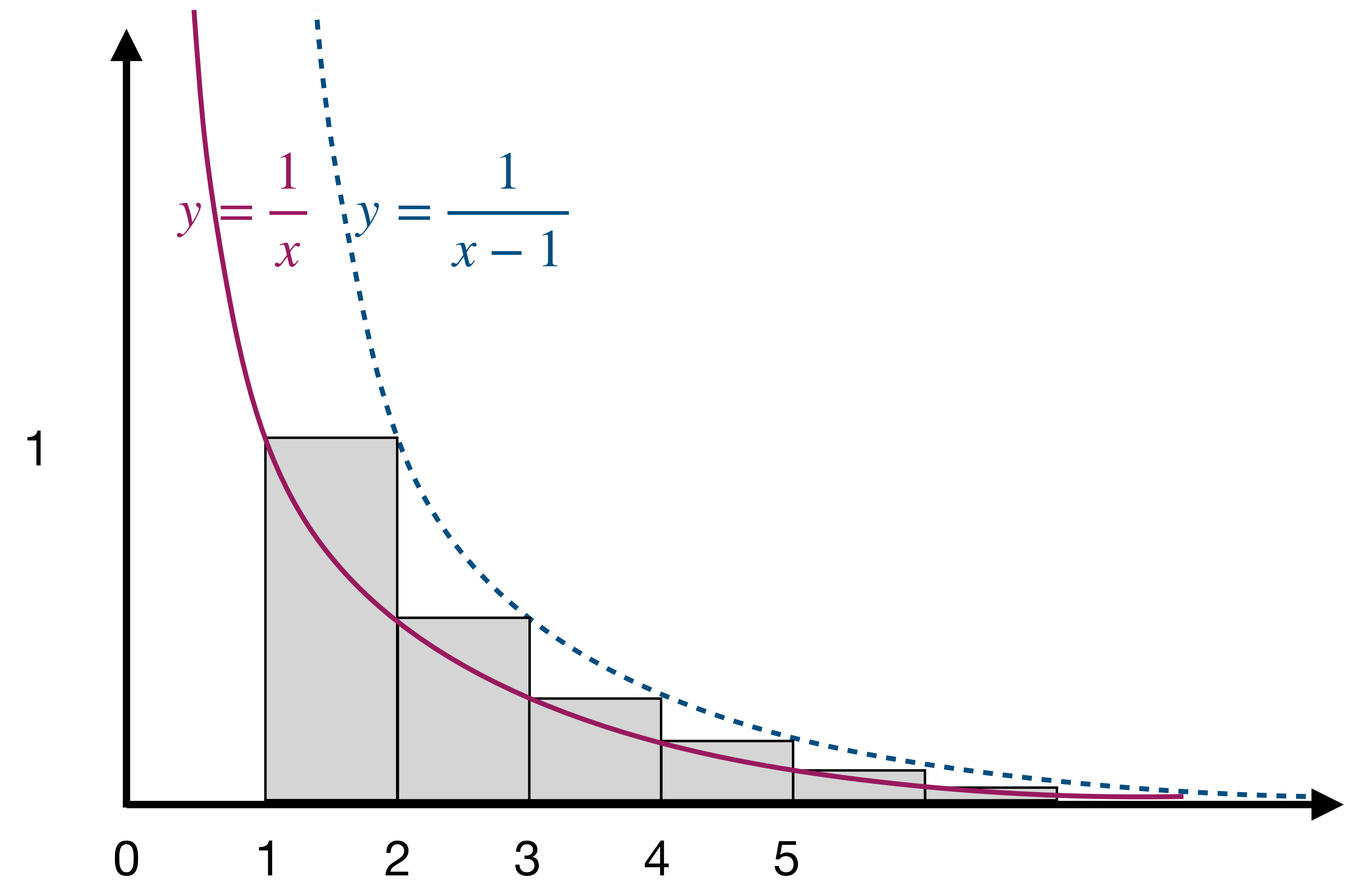
- $\mathbb{E}[X] = \mathbb{E}\left[\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} X_{ij}\right] = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} \mathbb{E}[X_{ij}] = \sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Pr(X_{ij}=1)$

Randomized QuickSort

- Let $Z_{ij} = \{z \mid z \in A, z_i \leq z \leq z_j\}$, where $i \leq j$, let $\hat{z}_{ij}$ be the first item in Z_{ij} that is chosen as a pivot. Then z_i are z_j compared iff $\hat{z}_{ij} = z_i$ or $\hat{z}_{ij} = z_j$. (Items from Z_{ij} stay in same split until some pivot is chosen from Z_{ij}).
- $Pr(X_{ij} = 1) = Pr(\hat{z}_{ij} = z_i) + Pr(\hat{z}_{ij} = z_j) = \dfrac{2}{j - i + 1}$
- $\mathbb{E}[X] = \displaystyle\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} \frac{2}{j - i + 1}$, let $k = j - i$, $\mathbb{E}[X] = \displaystyle\sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \frac{2}{k+1} < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k}$

Randomized QuickSort

- Harmonic series
  - $H_n = \sum_{k=1}^{n} \frac{1}{k}$
  - $\int_1^n \frac{1}{x} dx < \sum_{k=1}^{n} \frac{1}{k} < 1 + \int_2^n \frac{1}{x-1} dx$
  - $\ln n < \sum_{k=1}^{n} \frac{1}{k} < 1 + \ln n$



Randomized QuickSort

- Harmonic series
  - $H_n = \sum_{k=1}^{n} \frac{1}{k} \sim \ln n$
- Hence, $\mathbb{E}[X] < \sum_{i=1}^{n-1} \sum_{k=1}^{n} \frac{2}{k} < 2nH_n < 2n(1 + \ln n) = O(n \lg n)$
- Combined the fact that in the best case (balanced partition each time) randomized quick sort is $\Theta(n \lg n)$, the expected running time is $\Theta(n \lg n)$.
- In fact, runtime of `RndQuickSort` is $O(n \log n)$ with high probability!

A bit more on QuickSort

- What if there are many duplicates?
  - Maintain four regions as we go through the array

< pivot	= pivot	> pivot

  - End up with three regions (“<”, “=”, and “>”), and only recurse into two of them (“<” and “>”): the more the duplicates, the less to recurse, and the better the algorithm!

A bit more on QuickSort

- Stop recursion once the array is too small.
  - Recursion has overhead, `QuickSort` is slow on small arrays.
  - Usually using `InsertionSort` for ≈ 10 elements, resulting in fewer swaps, comparisons or other operations on such small arrays.
    - The ideal 'threshold' will vary based on the details of the specific implementation.

A bit more on QuickSort

- "Random pivot selection" and "Median of three" can be combined!
  - The expected number of comparisons needed to sort n elements with random pivot selection is $2n \ln n = \frac{2n}{log_2 e} \cdot log_2 n \approx 1.386 n \log_2 n$.
  - Combining "Median-of-three pivoting" (i.e., randomly selecting three elements and let the median of them to be the pivot) brings this down to about $1.188 n \log_2 n$, but at the expense of a three-percent increase in the expected number of **swaps**.
  - According to Bentley, Jon L.; McIlroy, M. Douglas (1993). "Engineering a sort function". Software: Practice and Experience. 23 (11): 1249–1265.

A bit more on QuickSort

- Multiple pivots?
  - Early studies do not give promising results, until Dual-Pivot variant proposed by Yaroslavskiy in 2009 seems slightly faster.

$< pivot_1$	$pivot_1 \leq . \leq pivot_2$	$> pivot_2$

  - This variant is used in Java for sorting. (Since Java 7.)
  - According to “Average Case Analysis of Java 7's Dual Pivot Quicksort”. (Best Paper of ESA 2012)

Summary on QuickSort

- A **widely-used** efficient sorting algorithm
- Easy to understand! (divide-and-conquer…)
- Moderately hard to implement correctly. (partition…)
- Harder to analyze. (randomization…)
- Challenging to optimize. (theory and practice…)

The $n \lg n$ sorting algorithms

- `QuickSort`, `MergeSort` and `HeapSort` are all with $O(n \lg n)$, which is better?
  - `HeapSort` is non-recursive, minimal auxiliary storage requirement (good for embedded system), but with poor **locality of reference**, the access of elements is not linear, resulting many caches being missed! It is the slowest among three algorithms
  - In most (not all) tests, `QuickSort` turns out to be **faster** than `MergeSort`. This is because although `QuickSort` performs 39% more comparisons than `MergeSort`, but much less movement (copies) of array elements.
  - `MergeSort` is a stable sorting, and can take advantage of partially pre-sorted input. Further, `MergeSort` is more efficient at handling slow-to-access sequential media.

External sorting

*External Sorting

- External sorting is required when the data being sorted do not fit into the main memory of a computing device and instead they must reside in the slower external memory, usually a disk drive.
- Since I/O is rather expensive (at the order of 1-10 milliseconds), the overall execution cost may be far dominated by the I/O, the target of algorithm design is to reduce I/Os.
- One challenge to previous internal sorting algorithms is that how to merge big files with small memory!

External merge problem

- **Input**: 2 **sorted** lists (with M and N pages)
- **Output**: 1 merged sorted list (with M+N pages)
- Can we efficiently (in terms of I/O) merge the two lists using a memory buffer of size at least 3?
    - Yes, and by using only 2(M+N) I/Os !

Key (Simple) Idea

- To find an element that is no larger than all elements in two lists, one only needs to compare minimum elements from each list

If:

$$A_1 \leq A_2 \leq \ldots \leq A_n$$

$$B_1 \leq B_2 \leq \ldots \leq B_m$$

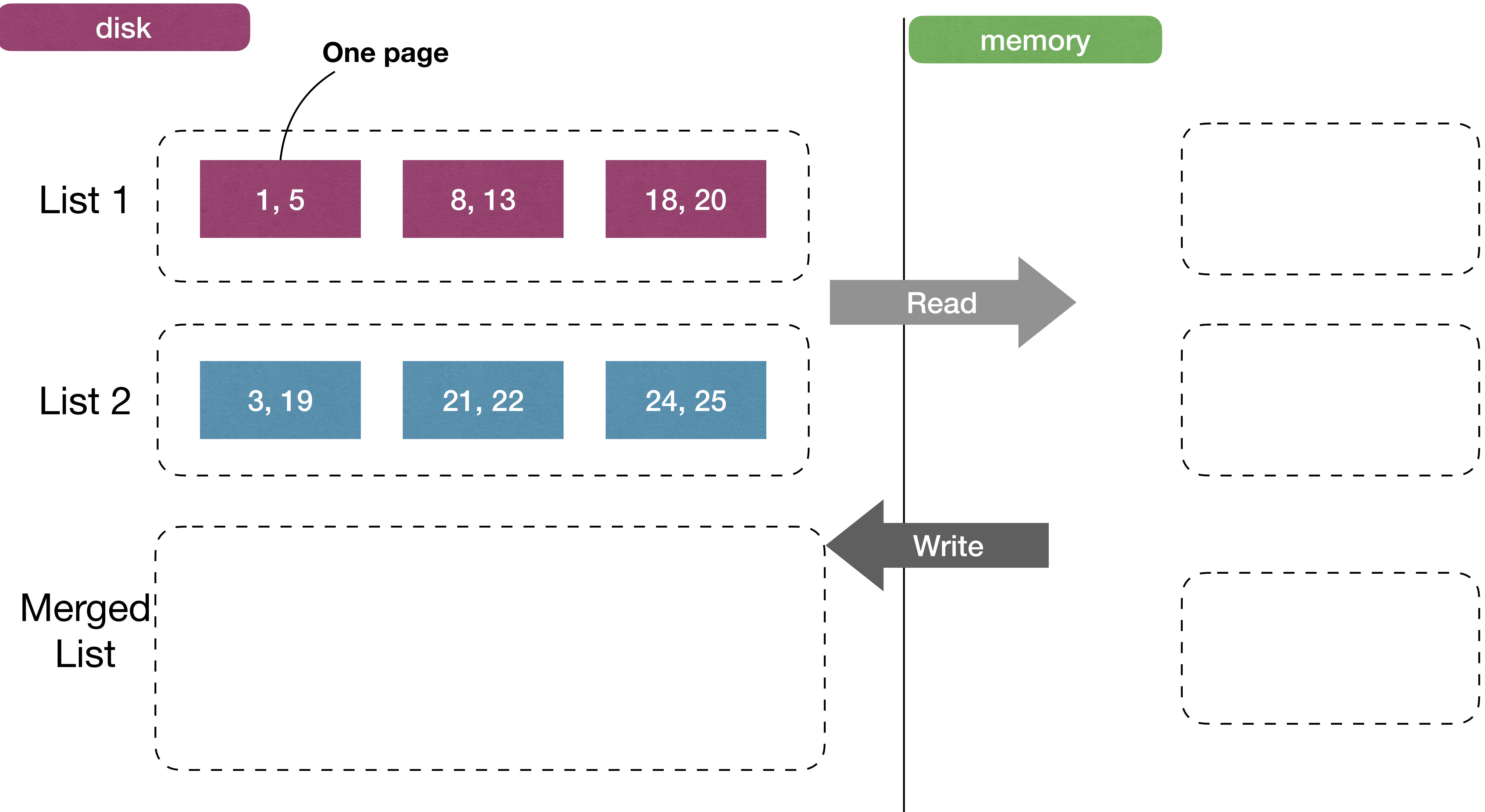
Then:

$$\min(A_1, B_1) \leq A_i, for\ 1 \leq i \leq n$$

$$\min(A_1, B_1) \leq B_j, for\ 1 \leq j \leq m$$

- Each time put the current minimum elements back to disk

External merge algorithm



External merge algorithm

External merge algorithm

智能软件与工程学院
School of Intelligent Software and Engineering

External merge algorithm

External merge algorithm

External merge algorithm

External merge algorithm

External merge algorithm

智能软件与工程学院
School of Intelligent Software and Engineering

External merge algorithm

External merge algorithm

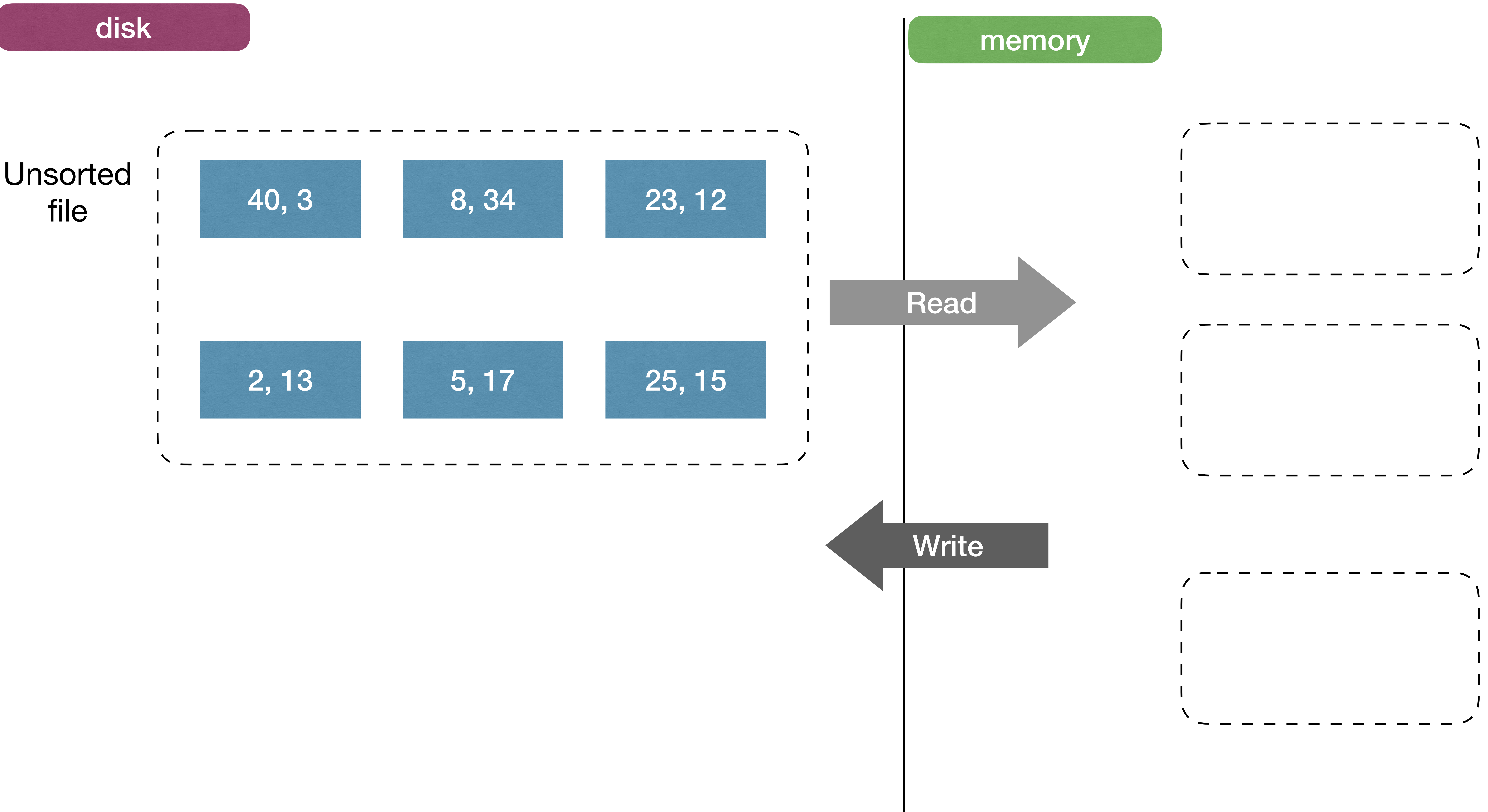
External merge cost

- We can merge 2 lists of arbitrary length with only 3 buffer pages.
  - I/O cost = $2(M+N)$
- When we have *B*+1 buffer pages, we can merge *B* lists with the same I/O cost

External merge sort

- How to deal with **unsorted** large files?
  - 1. Split into chunks small enough to sort in memory (“runs”)
  - 2. Merge pairs (or groups) of runs using the external merge algorithm
  - 3. Keep merging the resulting runs (each time = a “pass”) until left with one sorted file!

2-Way Sort


disk
memory
Unsorted file
40, 3
8, 34
23, 12
2, 13
5, 17
25, 15
Read
Write

智能软件与工程学院
School of Intelligent Software and Engineering

2-Way Sort

2-Way Sort


disk
memory
read each chunk in memory
40, 3
Read
2, 13
5, 17
25, 15
8, 34
Write
23, 12

2-Way Sort


disk
memory
sort in memory
3, 8
Read
2, 13
5, 17
25, 15
12, 23
Write
34, 40

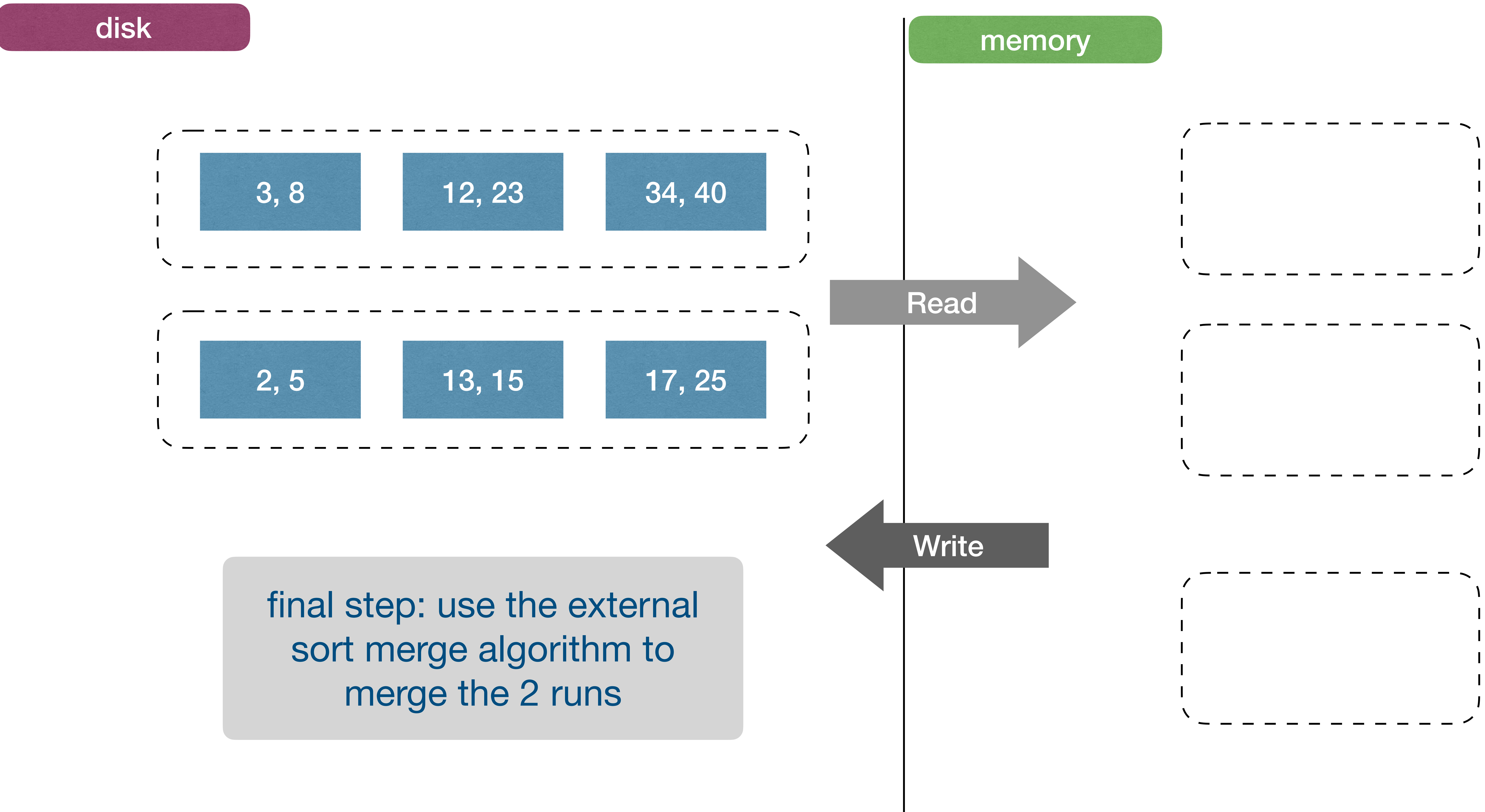
2-Way Sort


disk
memory
write back to disk
3, 8
12, 23
34, 40
2, 13
5, 17
25, 15
Read
Write
each sorted sub-file is called a run

2-Way Sort


disk
memory
same for the other chunk
3, 8
12, 23
34, 40
2, 5
13, 15
17, 25
Read
Write
Now we have 2 runs

2-Way Sort



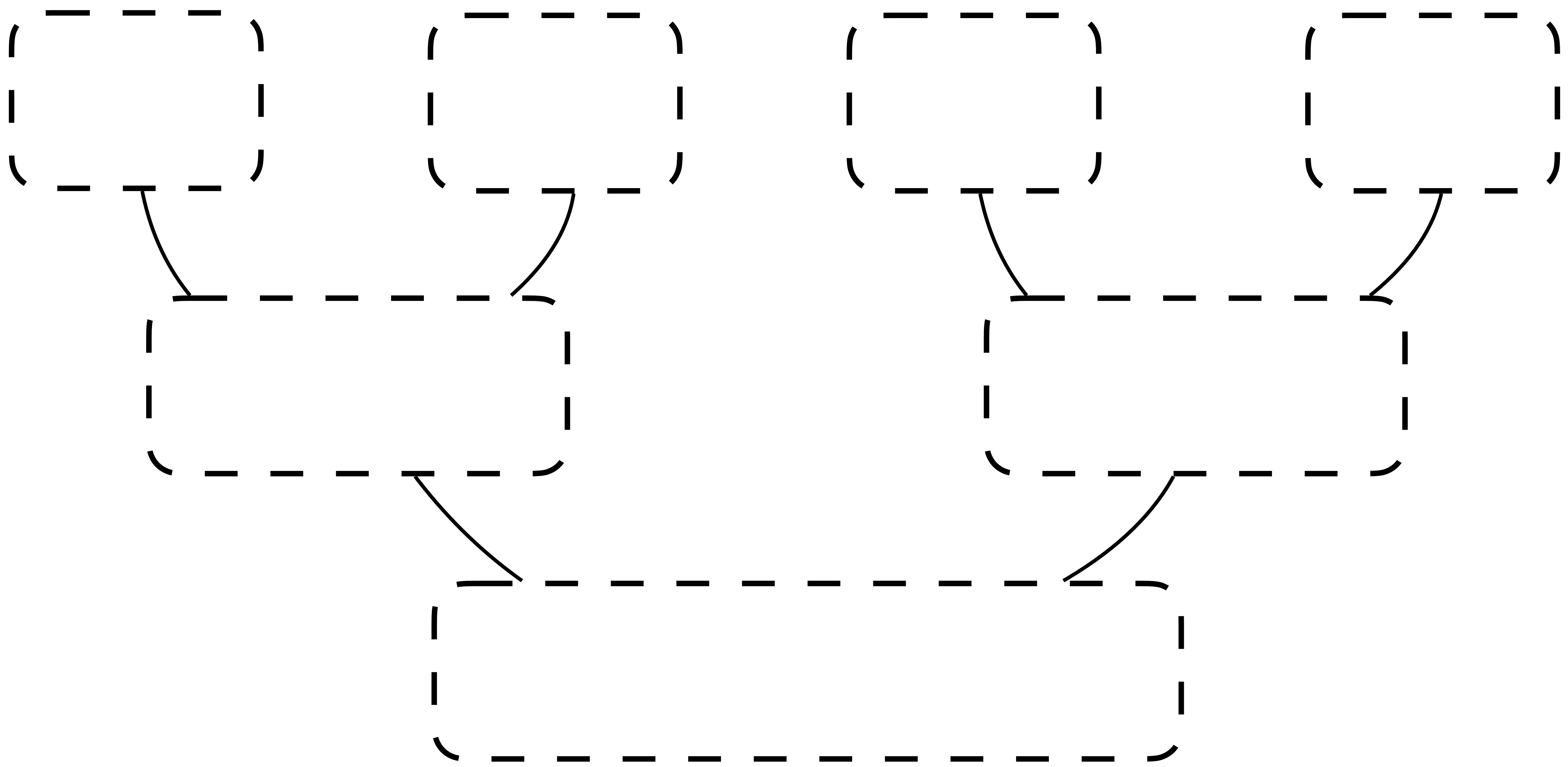
final step: use the external sort merge algorithm to merge the 2 runs

Calculating the I/O cost

- In our example there are 3 buffer pages, and 6 pages
- Pass 0: creating the runs
    - 1 read + 1 write for every page
    - total cost = $6 * (1 + 1) = 12$ I/Os
- Pass 1: external merge sort
    - total cost = $2 * (3 + 3) = 12$ I/Os
- So 24 I/Os in total

I/O Cost: Simplified Version

- Assume for now that we initially create N runs, each run consisting of a **single** page



pass 0: N runs, each 1 page
pass 1: merge into N/2 runs, each two pages
pass 2: merge into N/4 runs, each with 4 pages

- We need $\lceil \log_2 N \rceil + 1$ passes to sort the whole file, each pass needs $2N$ I/Os

- Total I/O cost = $2N(\lceil \log_2 N \rceil + 1)$

Can we do better?

- The 2-way merge algorithm only uses 3 buffer pages
- What if we have more available memory?
  - Use as much of the available memory as possible in every pass
  - Reducing the number of passes reduces I/O

External sort: I/O cost

- Suppose we have $B \geq 3$ buffer pages available

1. Increase length of initial runs

  - At the beginning, we can split the N pages into runs of length B and sort these in memory
  - IO cost:

$$2N(\lceil \log_2 N \rceil + 1) \rightarrow 2N(\lceil \log_2 \frac{N}{B} \rceil + 1)$$

Starting with runs of length 1 → **Starting with runs of length B**

External sort: I/O cost

- Suppose we have $B \geq 3$ buffer pages available

2. Perform a $(B-1)$—way merge.

  - On each pass, we can merge groups of $(B-1)$ runs at a time, instead of merging pairs of runs!
  - IO cost:

$$2N(\lceil \log_2 N \rceil + 1)$$

Starting with runs of length 1

$$2N(\lceil \log_2 \frac{N}{B} \rceil + 1)$$

Starting with runs of length B

$$2N(\lceil \log_{B-1} \frac{N}{B} \rceil + 1)$$

Performing $(B-1)$—way merge

Further reading

- [CLRS] Ch.7, Appendix C on probability theory
- [Weiss] Ch. 7 (7.4, 7.12)
- [Deng] Ch.12 (12.3)
- [TAOCP] Ch.5 (5.2.1 in vol. 3)