选择
Selection

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Order Statistics and Selection

- Given a set of n items, the i^{th} **order statistic**（顺序统计量）of it is the i^{th} smallest element of it.
  - Minimum, maximum, median, …
- The Selection Problem: given a set A of n distinct numbers and an integer i, find the i^{th} order statistic of A.

Find Min/Max

- So easy, sequential scan and keep *min/max* till now…

- Make n - 1 comparisons, but is this the best we can do?

- Yes! Otherwise at least two elements could be the minimum.
  - Initially each element could be the minimum.
  - An adversary answers queries like “compare x with y”.
  - Each comparison eliminates at most one element.

```
FindMin(A):
min := A[1]
for i := 2 to A.length
    if A[i] < min
        min := A[i]
return min
```

What if we want min *and* max?

- Go through the list twice, one for min and another for max.
- Can we do better? Surprisingly, yes!
  - Group items into pairs. (The first item becomes a “pair” if n is odd.)
  - For each of $\lceil n/2 \rceil$ pairs, find “local” min and max.

    $\lfloor n/2 \rfloor$ comparisons
  - Among $\lceil n/2 \rceil$ “local” min, find global min; similarly find global max.

    $\leq 2 \cdot \lfloor n/2 \rfloor$ comparisons

Total number of comparisons is at most $3 \cdot \lfloor n/2 \rfloor$

What if we want min *and* max?

- Is $3 \cdot \lfloor n/2 \rfloor$ the best we can do? Remarkably, yes!
  - An item has + mark if it can be max, and has - mark if it can be min.
  - Initially each item has both + and -.
  - An adversary answers queries like “compare x with y”.
  - The adversary can find input such that: at most $\lfloor n/2 \rfloor$ comparisons each removes two marks;
  - Every other comparison removes at most one mark.
  - In total need to remove $2n - 2$ marks.

So $\geq 2n - 2 - 2 * \lfloor n/2 \rfloor + \lfloor n/2 \rfloor = 2n - 2 - \lfloor n/2 \rfloor$ comparisons needed, which can be $3 \cdot \lfloor n/2 \rfloor$

General Selection Problem

- Find i^{th} smallest element (i.e., i^{th} order statistic)?
- Err… Sort them then return the i^{th} entry?
- Sure but this takes $\Omega(n \log n)$ time…

Can we be faster?

RndQuickSort(A):

if $A.size > 1$

$q := RandomPartition(A)$

$RndQuickSort(A[1, \ldots (q - 1)])$

$RndQuickSort(A[(q + 1), \ldots, n])$

General Selection Problem

- What if $i = q$?
  - $A[q]$ is what we need.
- What if $i < q$?
  - Find i^{th} order statistic in $A[1 \ldots (q - 1)]$.

Notice $A[1 \ldots (q\text{-}1)]$ contains the smallest q - 1 elements in A.

- What if $i > q$?

Find $(i - q)^{th}$ order statistic in $A[(q + 1) \ldots n]$.

RndQuickSort(A):

if $A.size > 1$

$\quad q := RandomPartition(A)$

$\quad RndQuickSort(A[1, \ldots (q - 1)])$

$\quad RndQuickSort(A[(q + 1), \ldots, n])$

This is Reduce-and-Conquer!

Randomized Selection

A Reduce-and-Conquer Algorithm

RndSelect(A, i):

if $A.size = 1$

    return $A[1]$

else

    $q := RandomPartition(A)$

    if $i = q$

        return $A[q]$

    else if $i < q$

        return $RndSelect(A[1 \ldots (q\text{-}1)], i)$

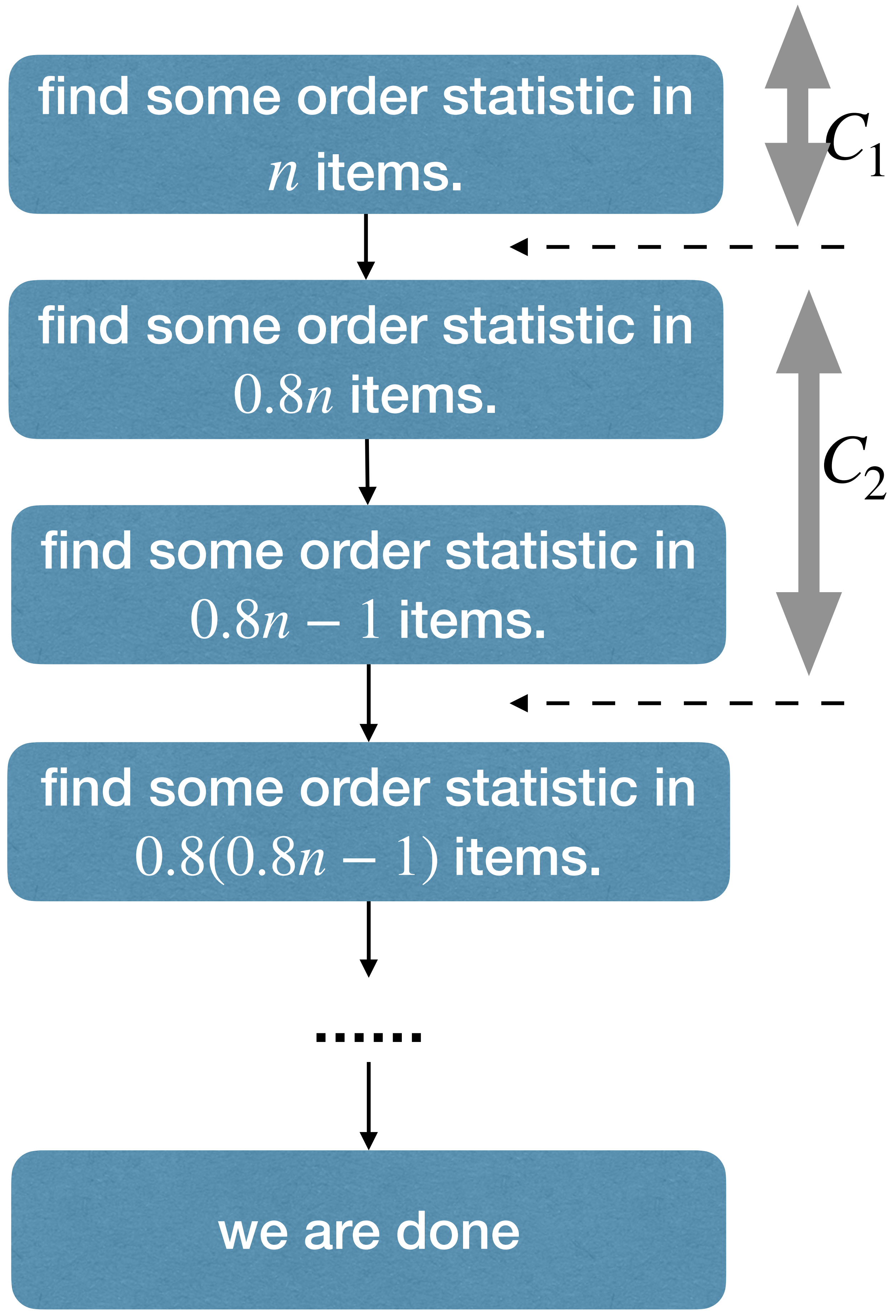
    else

        return $RndSelect(A[(q+1) \ldots A.size], i - q)$

- **Best-case** runtime? Choose the answer as the pivot in the first call (unlikely to happen).
  - $\Theta(n)$
- **Worst-case** runtime? Partition reduces array size by one each time (unlikely to happen).
  - $\geq cn + c(n-1) + \ldots + c(2) = \Theta(n^2)$
- What is the **average case**?

Average performance of Randomized Selection

- What's unlikely to happen is either get the exactly right pivot or reduces the size just by one. Instead, what's likely to happen is: partition process reduces problem size by a **constant** factor.
- Call a partition **good** if it reduces problem size to at most 0.8*input_size.
- Let the random variable C_i be the cost since the last good partition to the i^{th} good partition.
- At most $\log_{1.25} n$ good partitions can occur.
- $\mathbb{E}[C_i] \le \Theta(1) \cdot 0.8^{i-1} n$ **Why?**
- $\mathbb{E}[T(n)] \le \mathbb{E}\left[\sum_{i=1}^{log_{1.25}n} C_i\right] = \sum_{i=1}^{log_{1.25}n} \mathbb{E}[C_i] = O(n)$



RndQuickSort vs RndSelect

RndQuickSort(A):

if $A.size > 1$

  $q := RandomPartition(A)$

  $RndQuickSort(A[1, \ldots (q - 1)])$

  $RndQuickSort(A[(q + 1), \ldots, n])$

RndSelect(A, i):

if $A.size = 1$

  return $A[1]$

else

  $q := RandomPartition(A)$

  if $i = q$

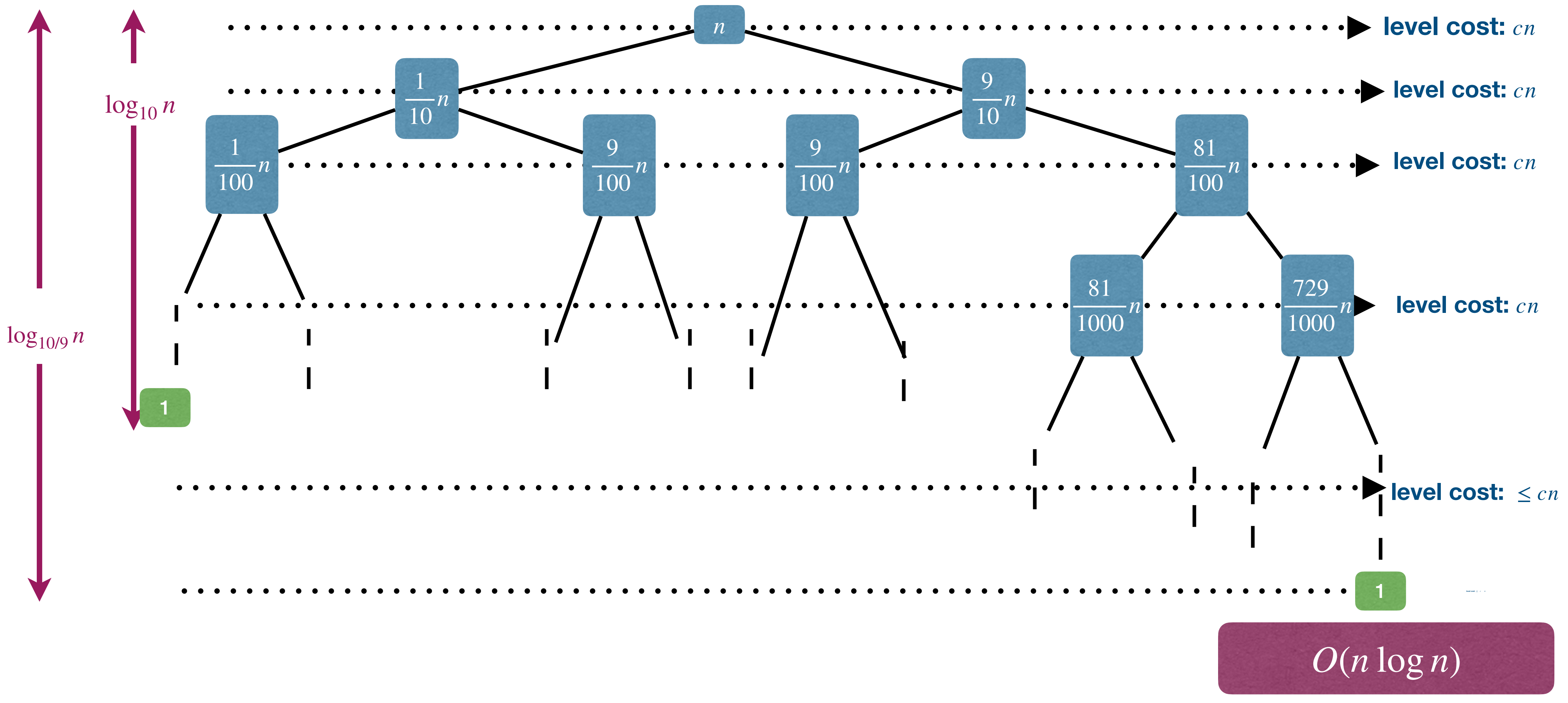
    return $A[q]$

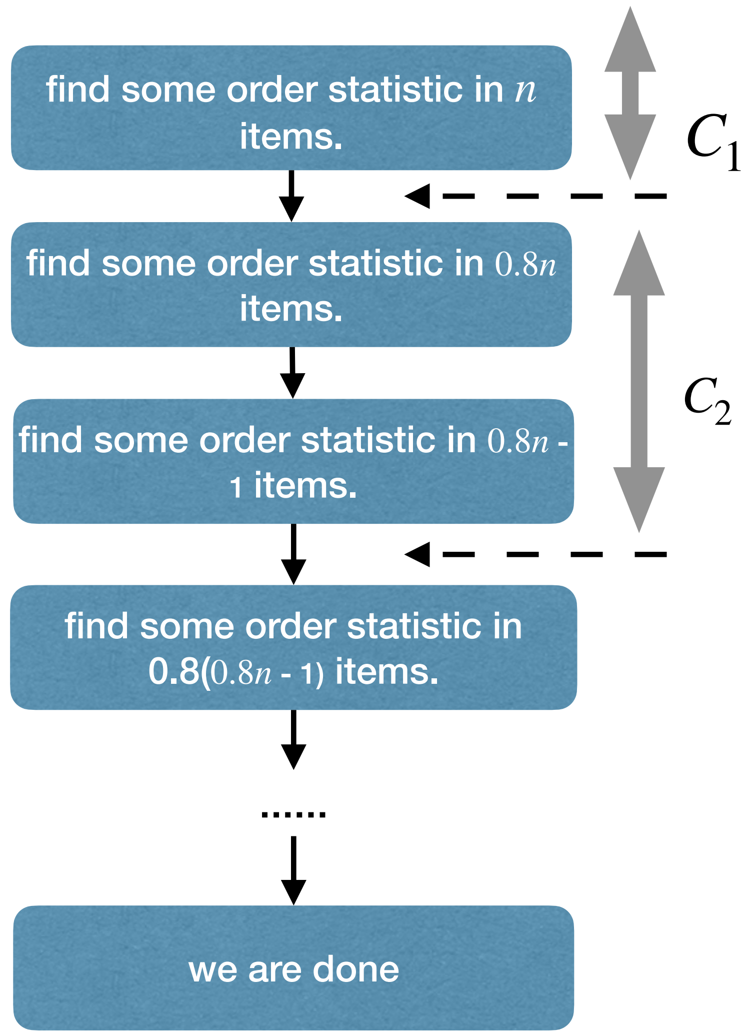
  else if $i < q$

    return $RndSelect(A[1 \ldots (q\text{-}1)], i)$

  else

    return $RndSelect(A[(q + 1) \ldots A.size], i - q)$





We are not done with selection...

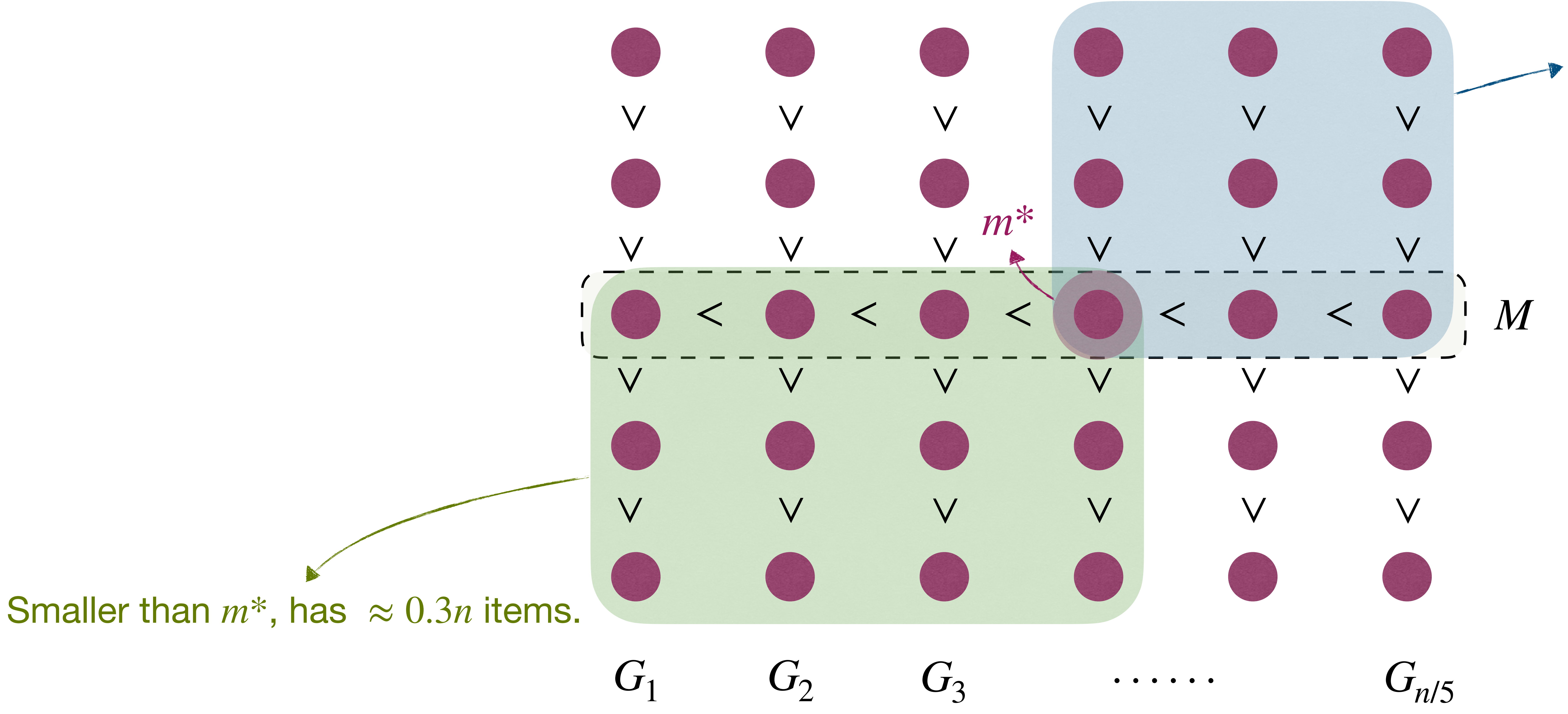
- Can we guarantee **worst-case** runtime of $O(n)$?
- The reason that `RndSelect` could be slow is that `RandomPartition` might return an **unbalanced** partition.
- Needs a partition procedure that guarantees to be **balanced**. (without using too much time; $O(n)$ time to be specific).

<u>RndSelect(A, i):</u>

if $A.size = 1$
  return $A[1]$
else
  $q := RandomPartition(A)$
  if $i = q$
    return $A[q]$
  else if $i < q$
    return $RndSelect(A[1 \ldots (q\text{-}1)], i)$
  else
    return $RndSelect(A[(q + 1) \ldots A.size], i - q)$

Median of medians

- Divide elements into $n/5$ groups, each containing 5 elements, call these groups $G_1, G_2, \ldots G_{n/5}$.
- Find the medians of these $n/5$ groups, let M be this set of medians.
- Find the median of M, call it m^*.



Partition using m^* as pivot is good: the smaller split has $\geq 0.3n$ items.

Finding median of medians

- Divide elements into $n/5$ groups, each containing 5 elements, call these groups $G_1, G_2, \ldots G_{n/5}$.

Trivial, $O(n)$ time

- Find the medians of these $n/5$ groups, let M be this set of medians.

Sort each group, then find the medians.
Cost is $(n/5) \cdot \Theta(1) = \Theta(n)$.

- Find the median of M, call it m^*.
  - Idea: Use `QuickSelect`, recursively.

Finding median of medians

QuickSelect(A, i):

if $A.size = 1$
  return $A[1]$
else
  $m := MedianOfMedians(A)$
  $q := PartitionWithPivot(A, m)$
  if $i = q$
    return $A[q]$
  else if $i < q$
    return $QuickSelect(A[1 \ldots (q\text{-}1)], i)$
  else
    return $QuickSelect(A[(q+1) \ldots A.size, i - q])$

$O(n)$

MedianOfMedians(A):

if $A.size = 1$
  return $A[1]$
$\langle G_1, G_2, \ldots G_{n/5} \rangle := CreateGroups(A)$
for $i := 1$ **to** $n/5$
  $Sort(G_i)$
$M := GetMediansFromSortedGroups(G_1, G_2, \ldots G_{n/5})$
return $QuickSelect(M, (n/5)/2)$

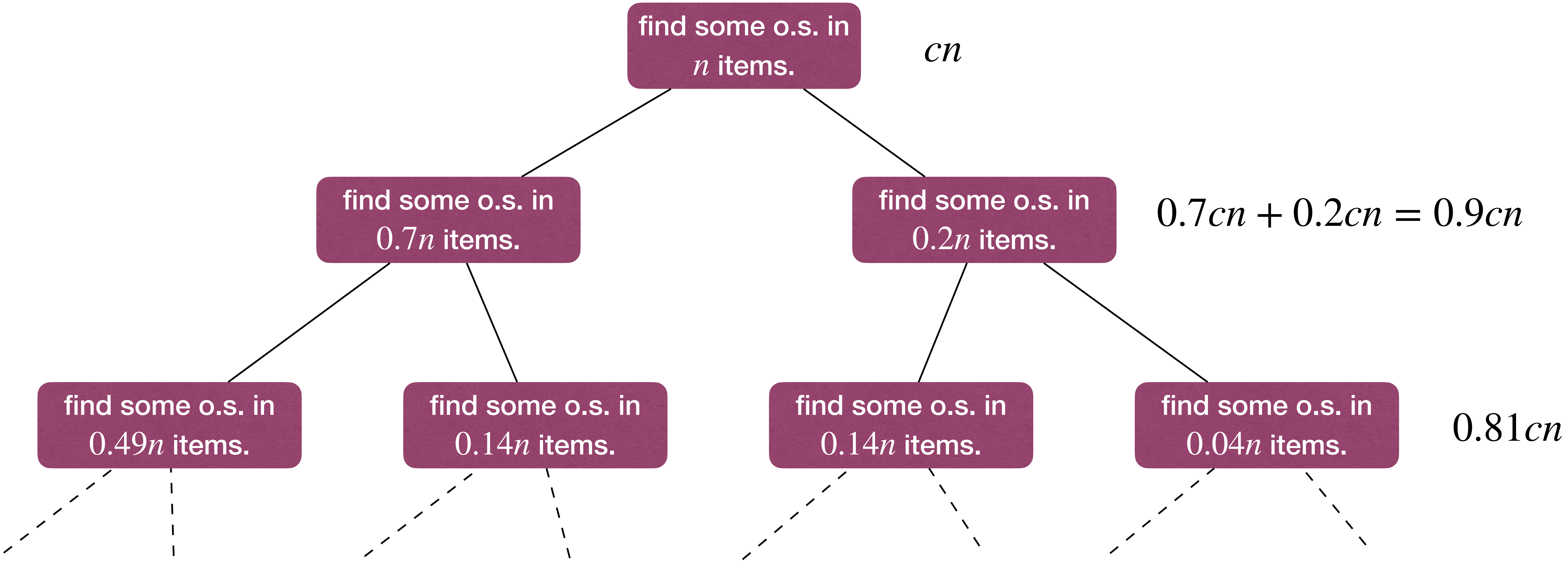
$T(0.2n)$

M is $\frac{1}{5}$ of A !

$T(0.7n)$

$$T(n) \leq T(0.7n) + T(0.2n) + O(n)$$

Time complexity



$O(n)$ in total

Time complexity

QuickSelect(A, i):

if $A.size = 1$
    return $A[1]$
else
    $m := MedianOfMedians(A)$
    $q := PartitionWithPivot(A, m)$
    if $i = q$
        return $A[q]$
    else if $i < q$
        return $QuickSelect(A[1 \ldots (q\text{-}1)], \mathrm{i})$
    else
        return $QuickSelect(A[(q{+}1) \ldots A.size], i - q])$

MedianOfMedians(A):

if $A.size = 1$
    return $A[1]$
$\langle G_1, G_2, \ldots G_{n/5} \rangle := CreateGroups(A)$
for $i := 1$ **to** $n/5$
    $Sort(G_i)$
$M := GetMediansFromSortedGroups(G_1, G_2, \ldots G_{n/5})$
return $QuickSelect(M, (n/5)/2)$

- $T(n) \leq T(0.7n) + T(0.2n) + O(n)$
- $T(n) = O(n)$

You can verify this by the substitution method. (I.e., assume $T(n) \leq cn$ and then verify.)

Complexity of general selection

- `QuickSelect` uses $O(n)$ time/comparisons.
- Solving general selection needs at least n - 1 comparisons.
  - Since finding min/max needs at least n - 1 comparisons.
- So the lower and upper bounds match asymptotically.
- But if we care about constants, needs (much) more efforts.

Further reading

- [CLRS] Ch.9