搜索树
Search Trees

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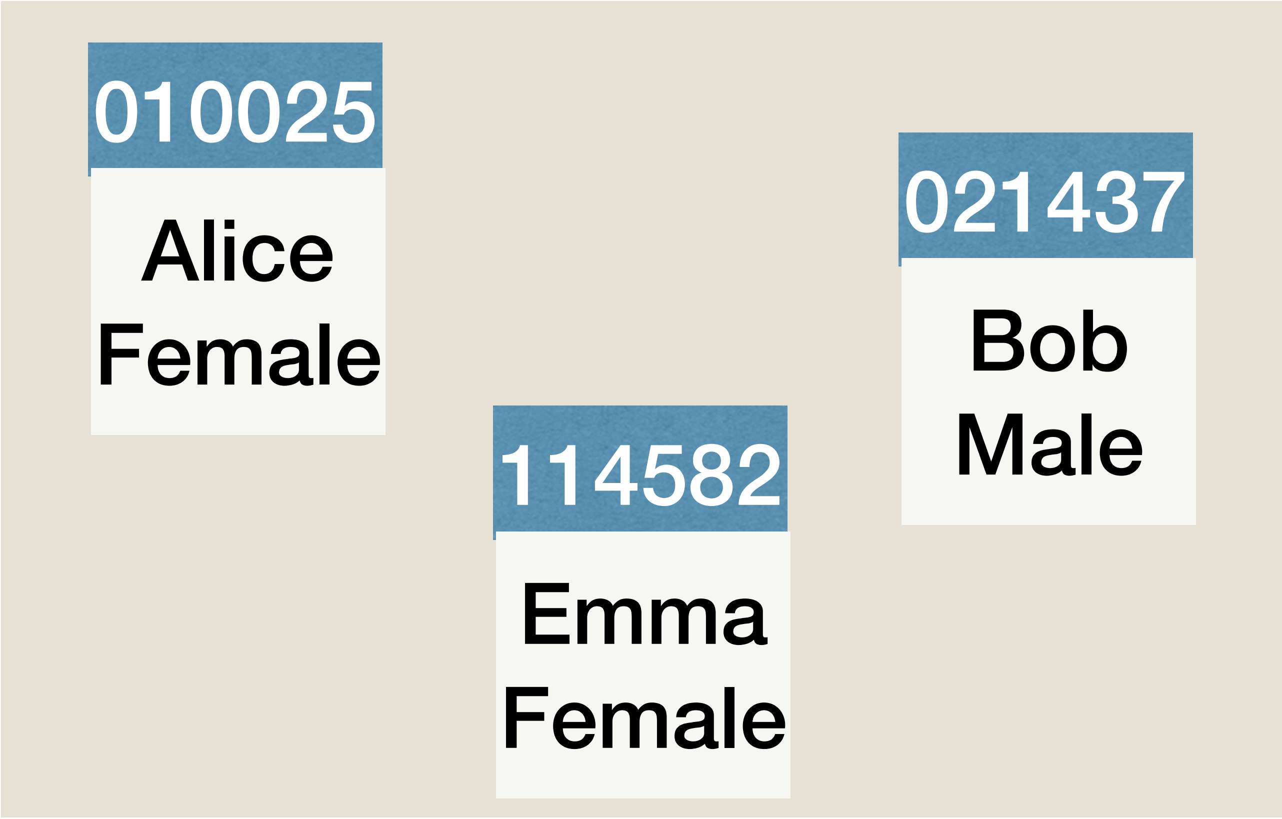
The slides are mainly adapted from the original ones shared by Chaodong Zheng and Kevin Wayne. Thanks for their supports!

The Dictionary Abstract Data Type

- A **Dictionary** (also **symbol-table**, **relation**, **map**) ADT is used to represent a **set** of elements with (usually distinct) **key** values.
  - Each element has a `key` field and a `data` field.



- Operations the Dictionary ADT should support:
  - **`Search(S,k)`**: Find an element in `S` with key value `k`.
  - **`Insert(S,x)`**: Add `x` to `S`. (What if element with same key exists?)
  - **`Remove(S,x)`**: Remove element `x` from `S`, assuming `x` is in `S`.
  - **`Remove(S,k)`**: Remove element with key value `k` from `S`.

Convention: the new value replaces the old one

The Dictionary Abstract Data Type

- In typical applications, keys are from an ordered universe (**Ordered Dictionary**):
  - `Min(S)` and `Max(S)`: Find the element in `S` with minimum/maximum key.
  - `Successor(S,x)` or `Successor(S,k)`:
    - Find smallest element in `S` that is larger than `x.key` (or key `k`).
  - `Predecessor(S,x)` or `Predecessor(S,k)`:
    - Find largest element in `S` that is smaller than `x.key` (or key `k`).

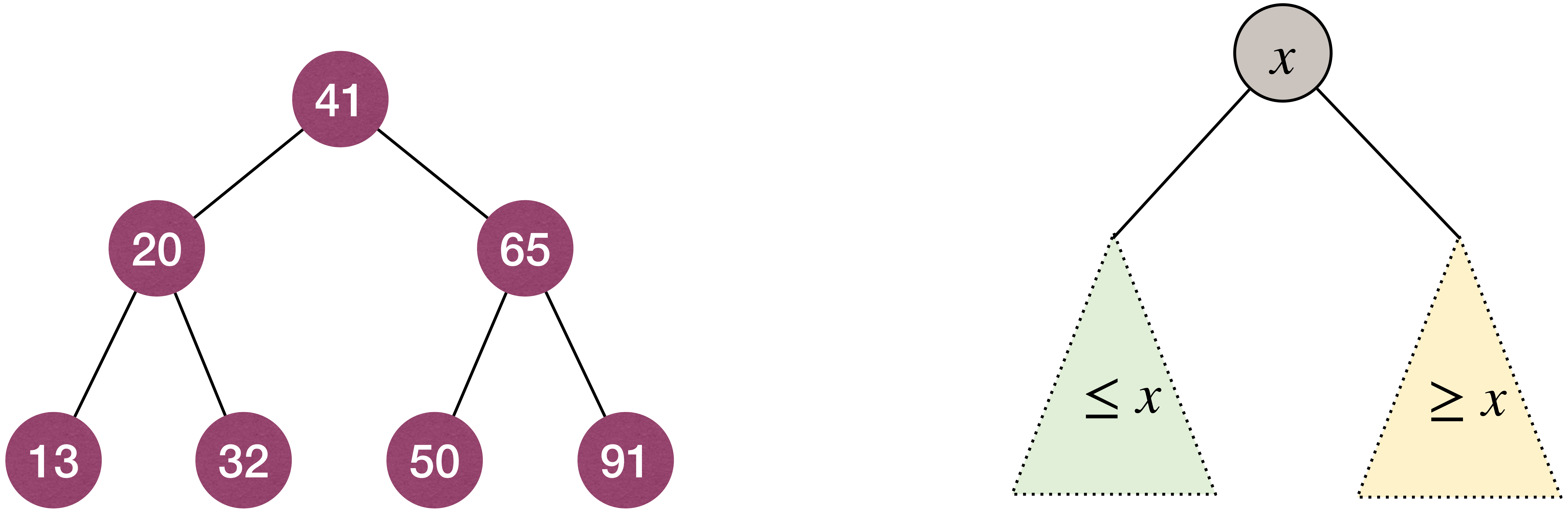
Efficient implementation of Ordered Dictionary

	`Search(S,k)`	`Insert(S,x)`	`Remove(S,x)`
`SimpleArray`	$O(n)$	$O(1)$	$O(n)$
`SimpleLinkedList`	$O(n)$	$O(1)$	$O(1)$
`SortedArray`	$O(\log n)$	$O(n)$	$O(n)$
`SortedLinkedList`	$O(n)$	$O(n)$	$O(1)$
`BinaryHeap`	$O(n)$	$O(\log n)$	$O(\log n)$

- Data structure implementing all these operations efficiently?
  - Efficient means within $O(\log n)$ time.

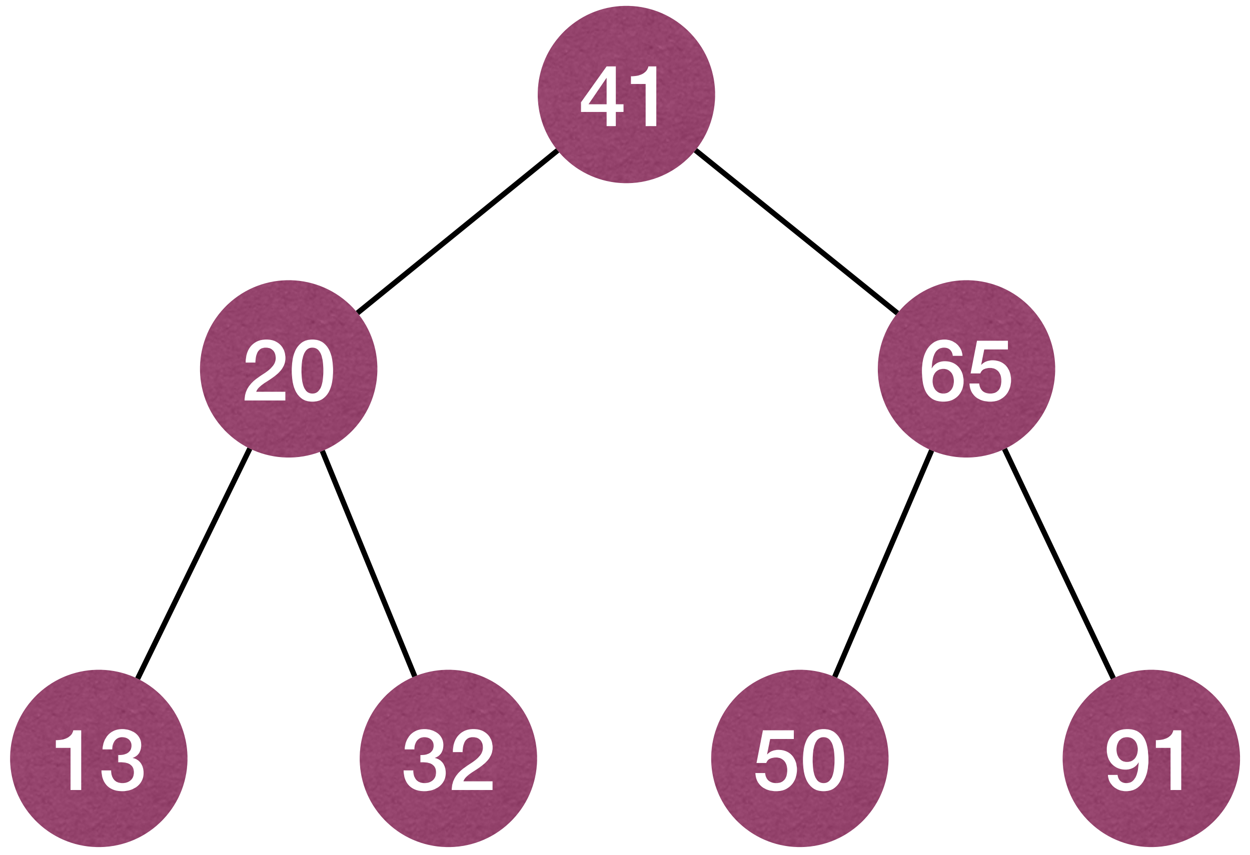
Binary Search Tree (BST)

- A **binary search tree (BST)** is a binary tree in which each node stores an element, and satisfies the **binary-search-tree property (BST property)**:
  - For every node x in the tree, if y is in the left subtree of x, then $y.key \leq x.key$; if y is in the right subtree of x, then $y.key \geq x.key$.



Binary Search Tree (BST)

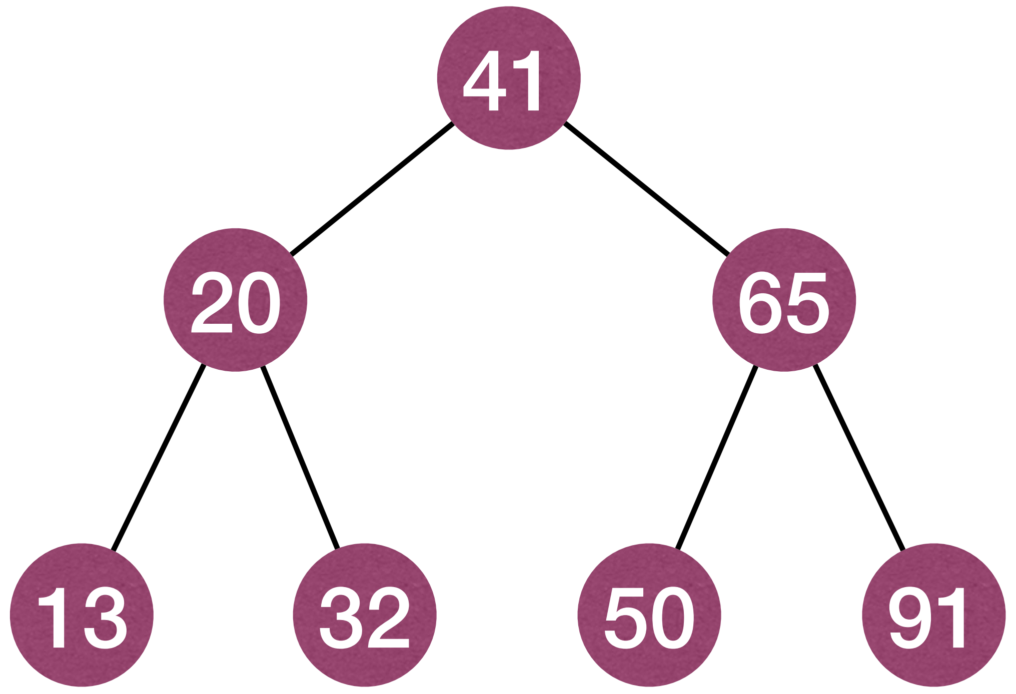
- Given a BST T, let S be the set of elements stored in T, what is the sequence of the in-order traversal of T?
  - Elements of S in ascending order!

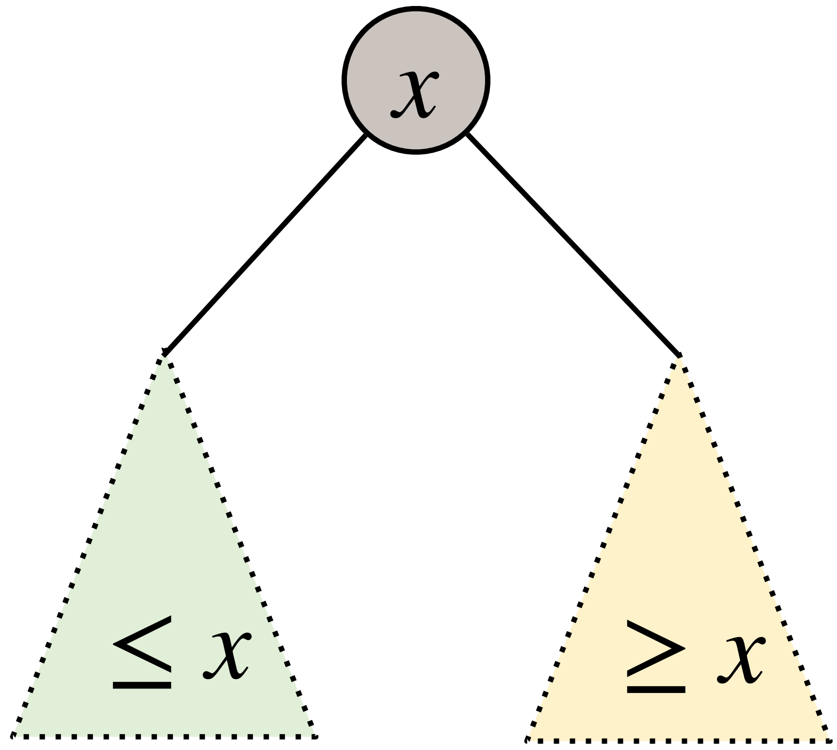


Inorder traversal: 13, 20, 32, 41, 50, 65, 91

Search in BST

- Given a BST root x and key k, find an element with key k?
  - If $x.key = k$ then return x and we are done!
  - If $x.key > k$ then **recurse** into the BST rooted at $x.left$.
  - If $x.key < k$ then **recurse** into the BST rooted at $x.right$.





BSTSearch(x,k):

if $x = NULL$ **or** $x.key = k$
    return x
else if $x.key > k$
    return $BSTSearch(x.left, k)$
else
    return $BSTSearch(x.right, k)$

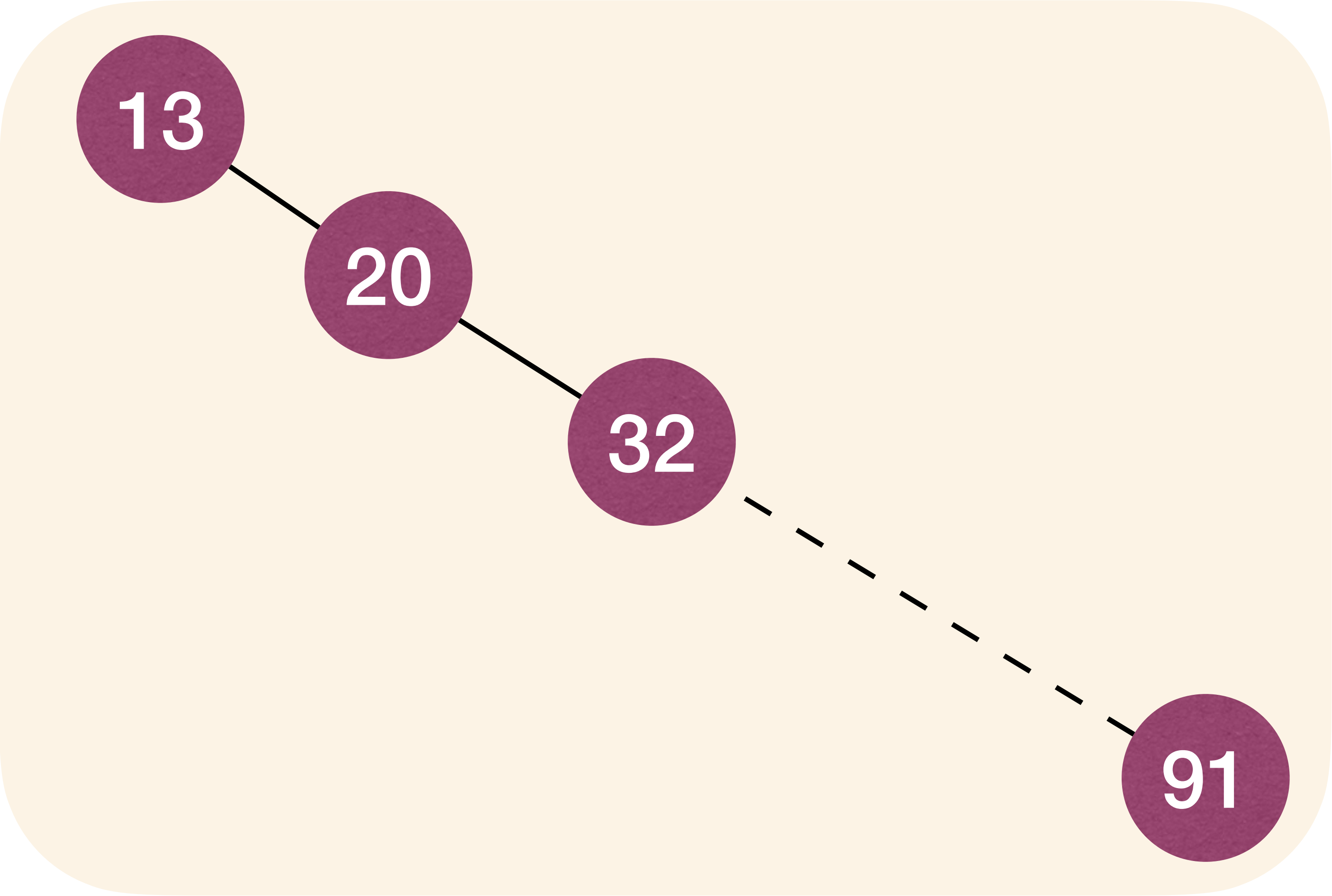
tail recursion → iterative version

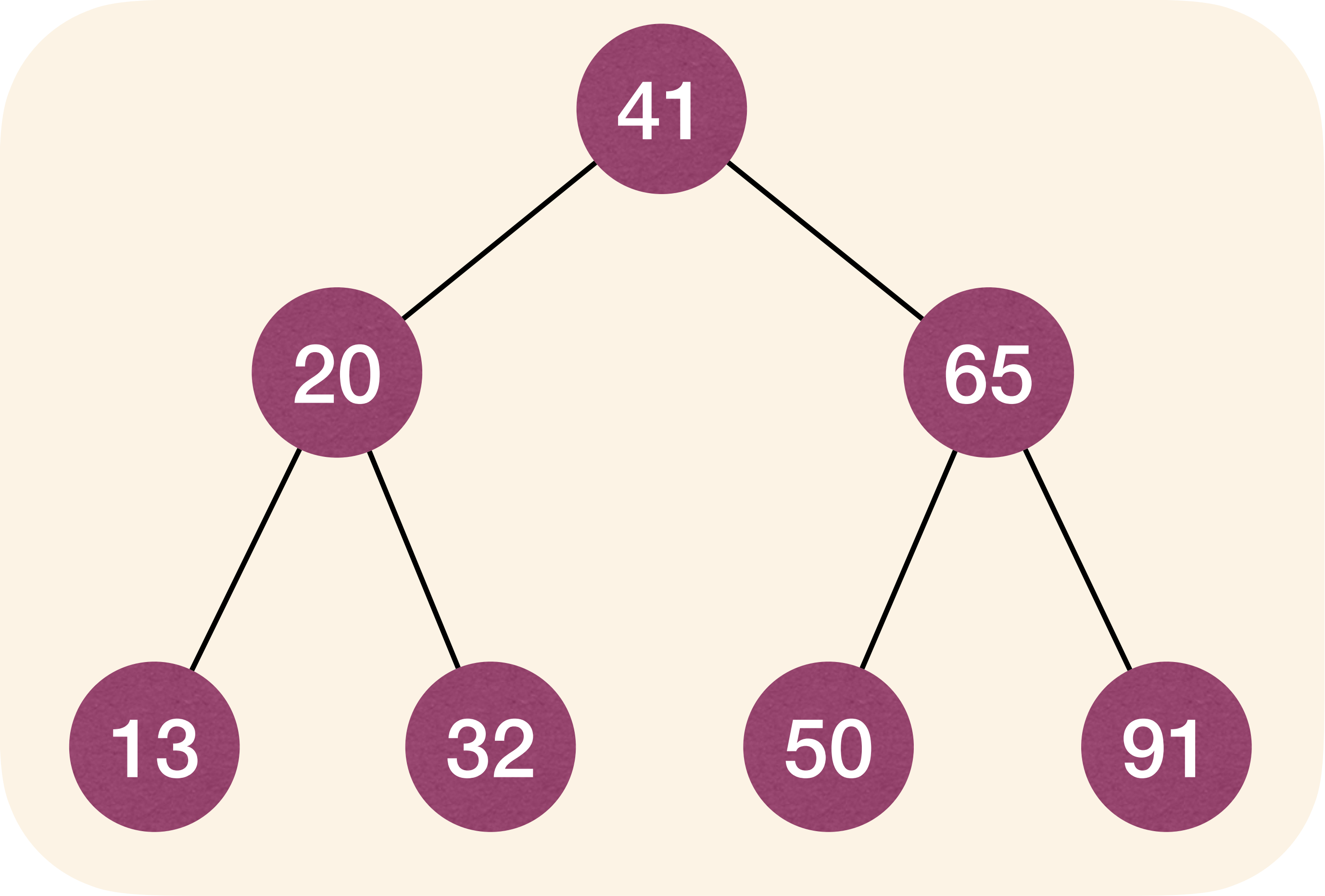
BSTSearchIter(x,k):

while x != $NULL$ **and** $x.key$!= k
    if $x.key > k$
        $x = x.left$
    else
        $x = x.right$
return x

Complexity of Search in BST

- Worst-case time complexity of Search operation?
  - $\Theta(h)$ where h is the height of the BST.
- How large can h be in an n-node BST?
  - $\Theta(n)$, when the BST is like a "path".
- How small can h be in an n-node BST?
  - $\Theta(\log n)$, when the BST is "well balanced".

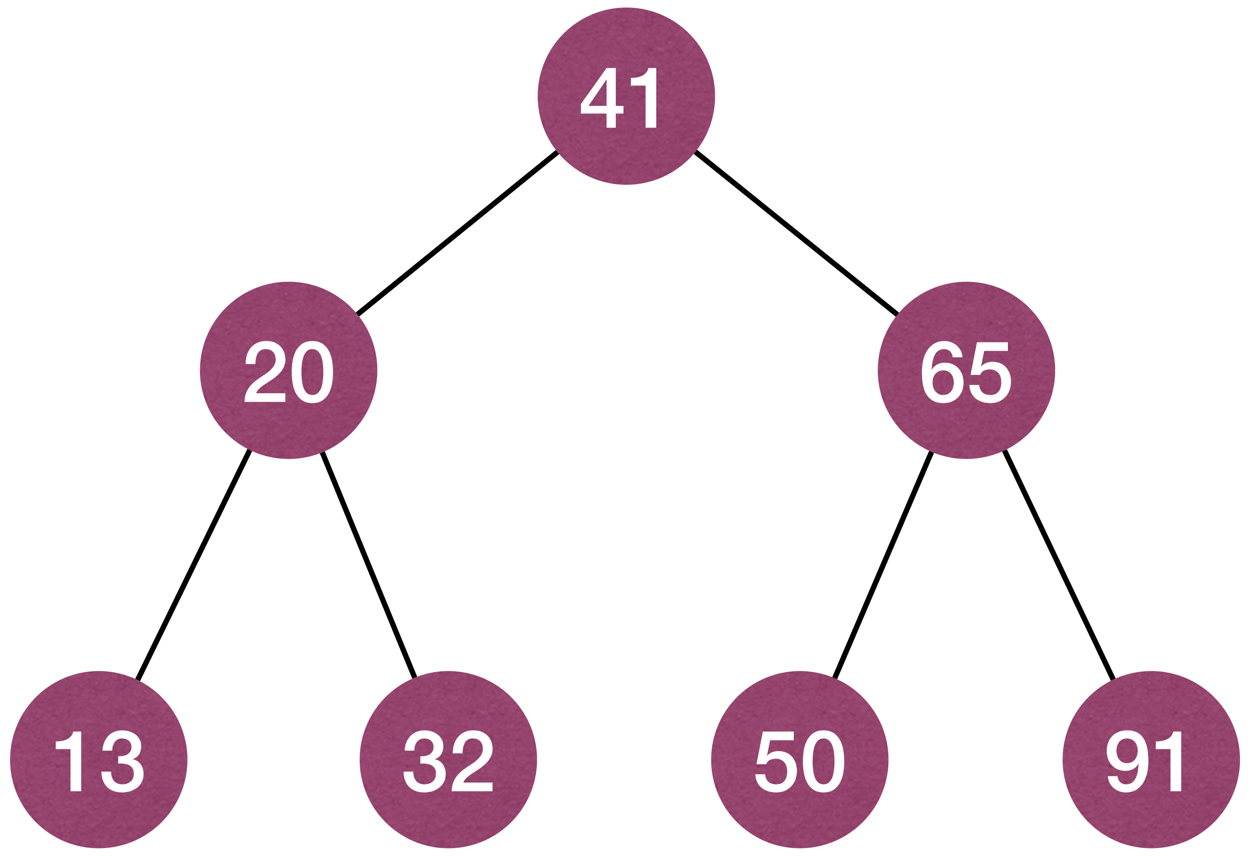




Height of the BST affects the efficiency of Search

Min and Max in BST

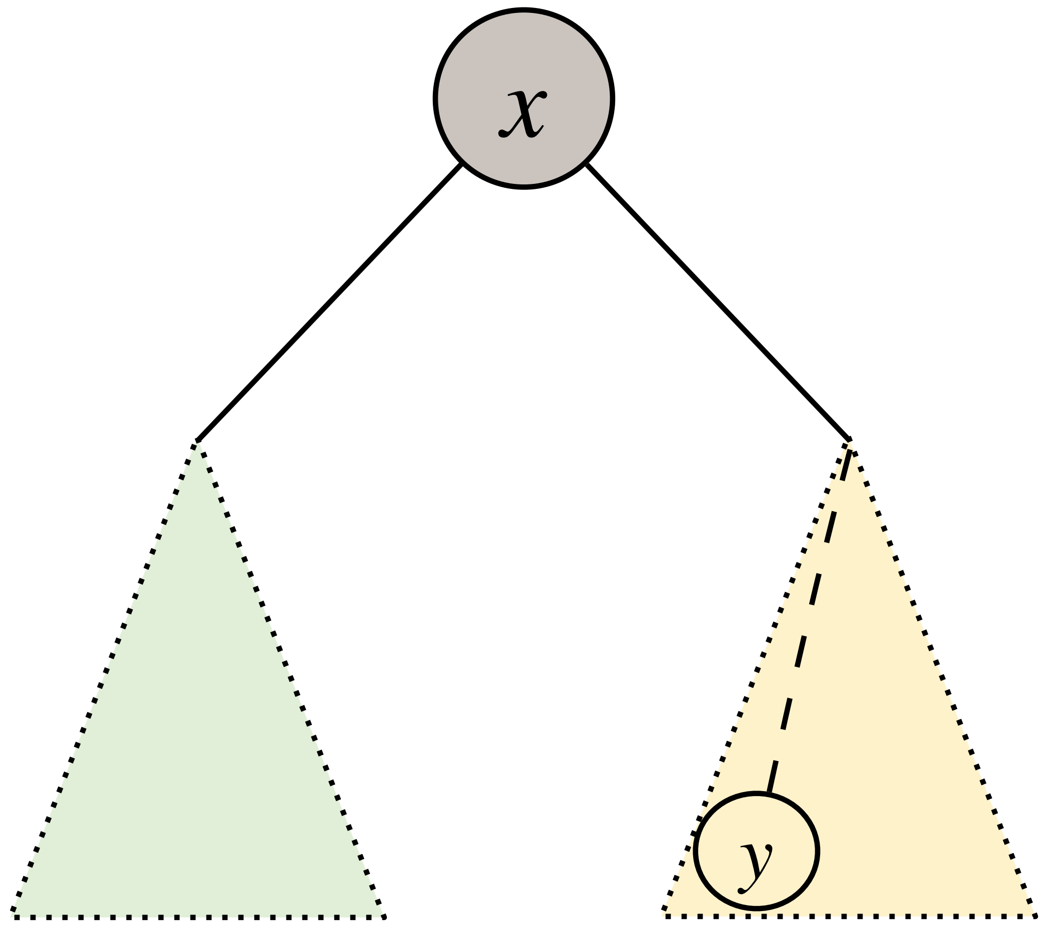
- How to find a minimum element in a BST?
  - Keep going left until a node without left child.
- How to find a maximum element in a BST?
  - Keep going right until a node without right child.
- Time complexity of Min and Max operation?
  - $\Theta(h)$ in the worst-case where h is height.



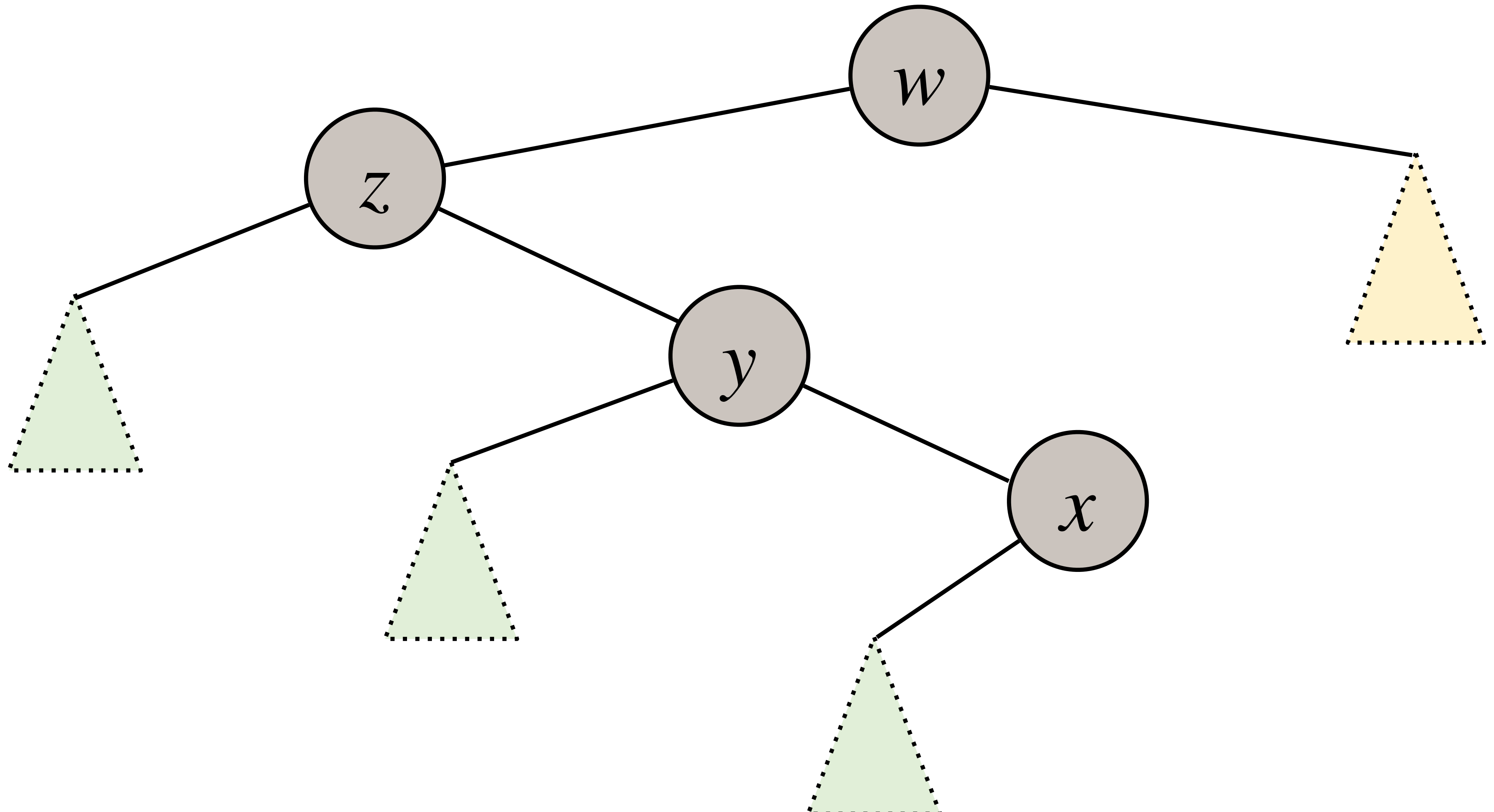
Successor in BST

- **BSTSuccessor(x)**: Find the smallest element in the BST with key value larger than $x.key$.
- In-order traversal of BST lists the elements in sorted order. Where in the tree does the element following x reside?

If the right subtree rooted at x is non-empty:
The minimum element in BST rooted at $x.right$ is what we want.



Otherwise:
The nearest ancestor of x whose left child is also ancestor of x.



Successor in BST

- `BSTSuccessor(x)`: Find the smallest element in the BST with key value larger than $x.key$.

- In-order traversal of BST lists the elements in sorted order.

BSTSuccessor(x,k):

```
if x.right != NULL
    return BSTMin(x.right)
y := x.parent
while y != NULL and y.right = x
    x := y
    y := y.parent
return y
```

- Time complexity of **BSTSuccessor**?
  - $\Theta(h)$ in the worst-case where h is the height.
- **BSTPredecessor** can be designed and analyzed similarly.

Operations change BST

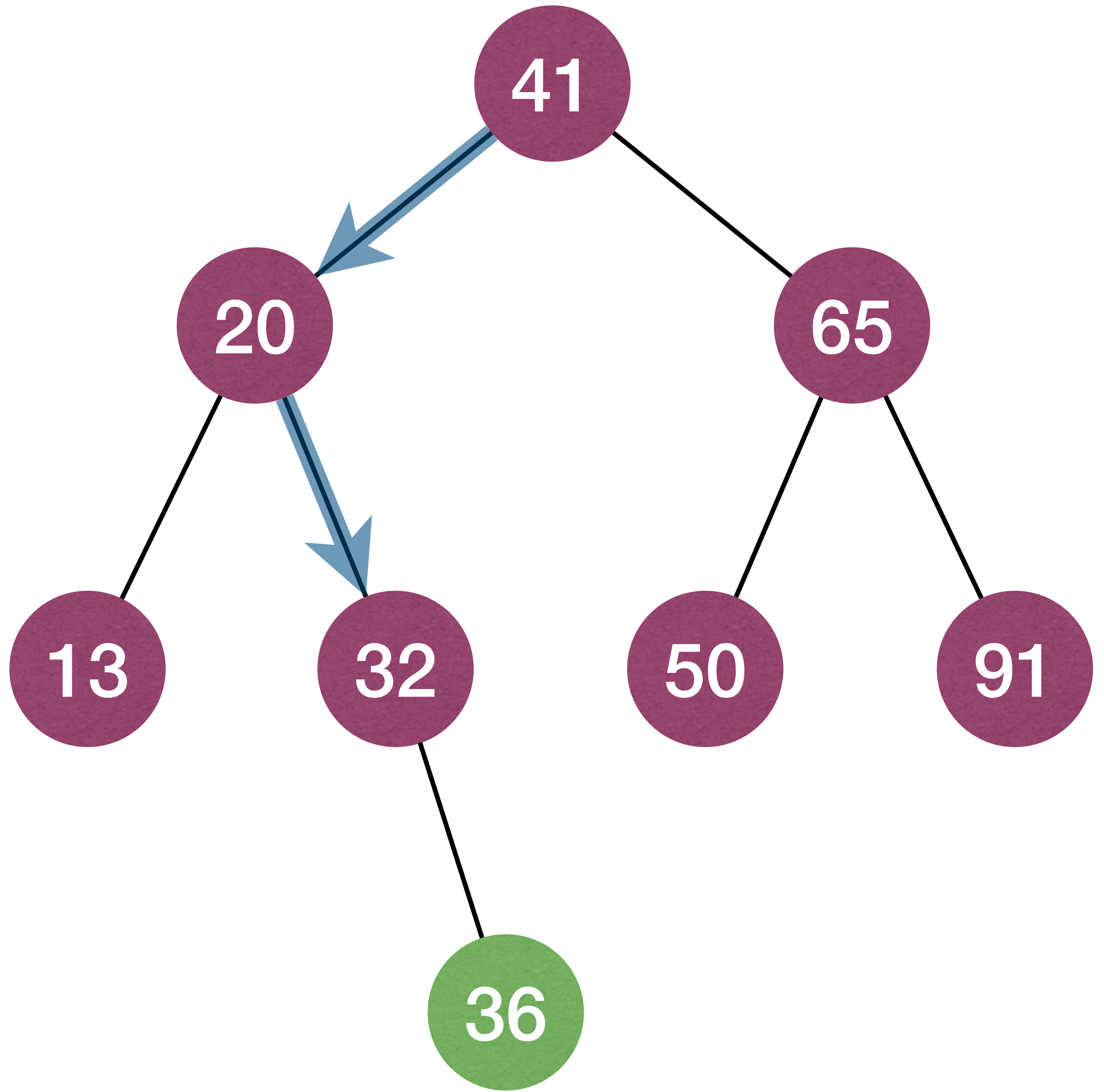
- So far we've seen operations that do not change the BST.
  - **Search**, **Min/Max**, **Successor/Predecessor**.
- How about operations that will change the BST?
  - **Insert** and **Remove**.

Insert in BST

- `BSTInsert(T,z)`: Add z to BST T. Notice, insertion should not break the BST property.

- Just like doing a search in T with key $z.key$. This search will fail and end at a leaf y. Insert z as left or right child of y.

Example: Insert element with key 36



Why above procedure is correct?

Insert in BST

- `BSTInsert(T,z)`: Add z to BST T. Notice, insertion should not break the BST property.

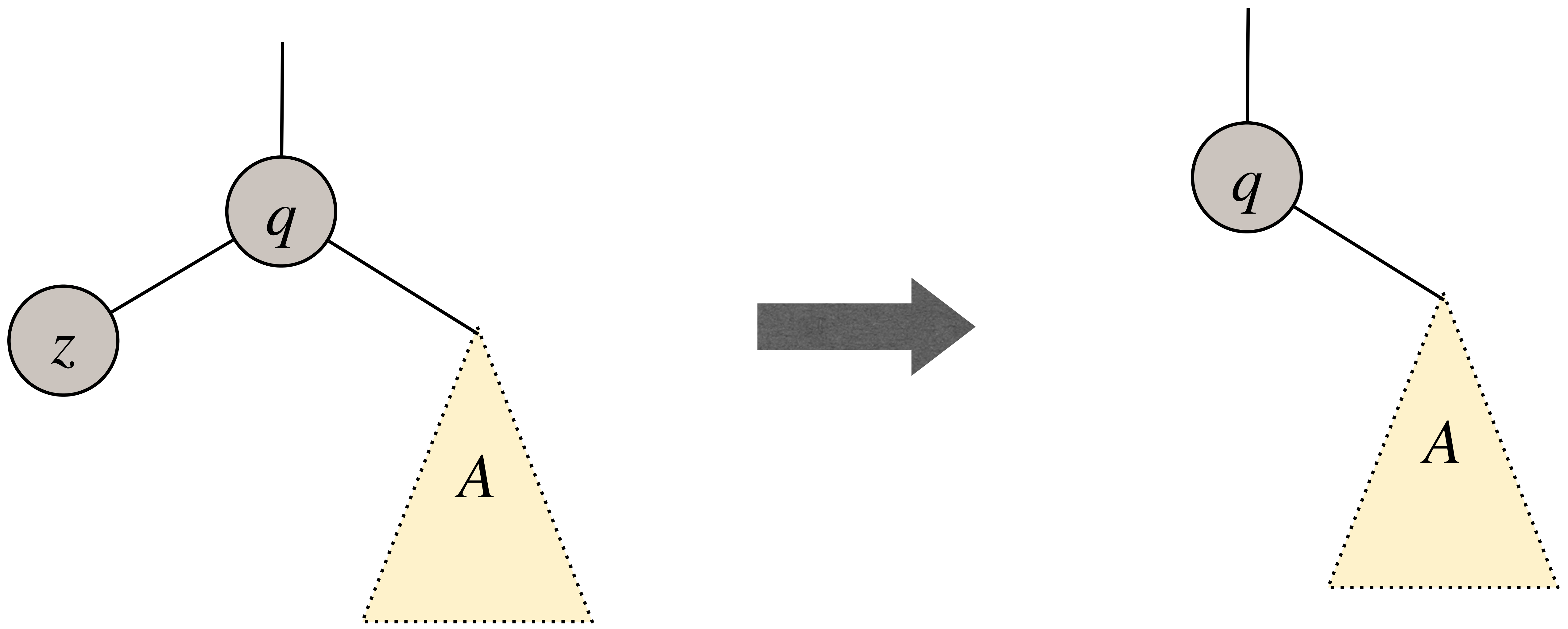
- Just like doing a search in T with key $z.key$. This search will fail and end at a leaf y. Insert z as left or right child of y.

- Time complexity of the Insert operation?
    - $\Theta(h)$ in the worst-case where h is the height of T.

Remove in BST

- **`BSTRemove(T,z)`**: Remove element z from T. Notice, removal should not break the BST property.

- **Case 1**: z has no child.
  - Easy, simply remove z from the BST tree



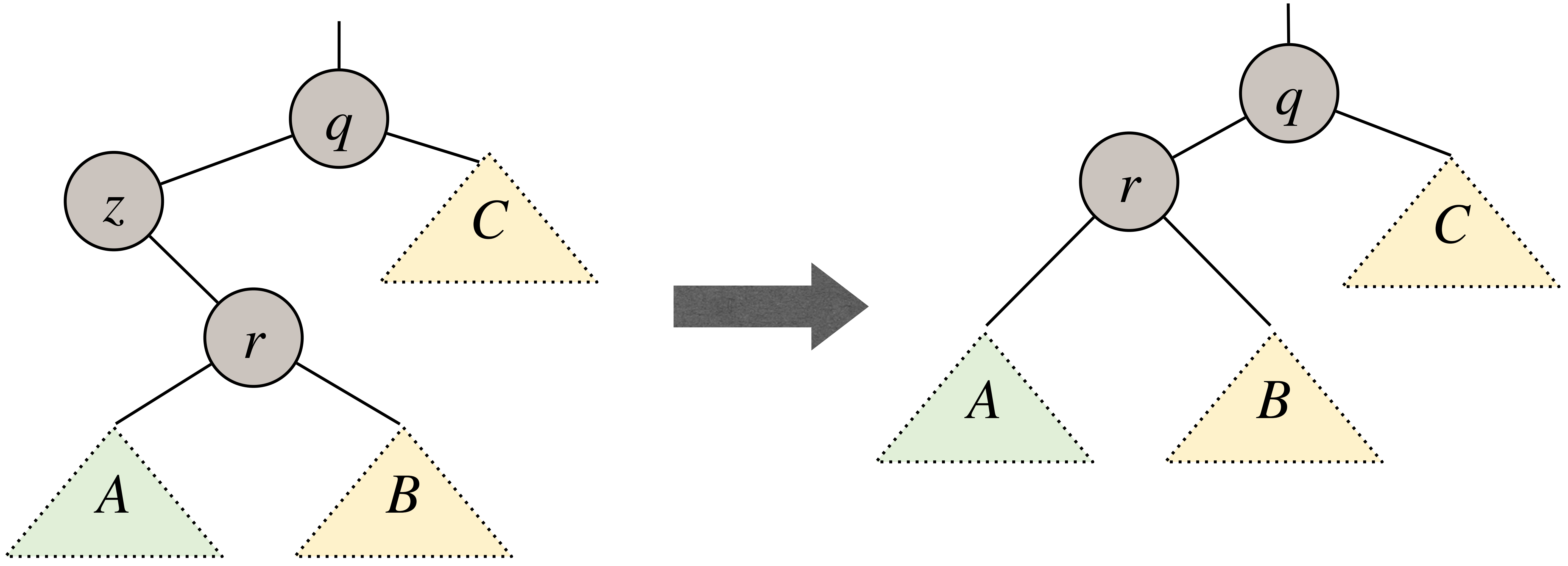
$\ldots, z, q, A, \ldots$ $\quad\quad$ $\ldots, q, A, \ldots$

Remove in BST

- `BSTRemove(T,z)`: Remove element z from T. Notice, removal should not break the BST property.

- **Case 2**: z has one single child.

  - Elevate subtree rooted at z's single child to take z's position.

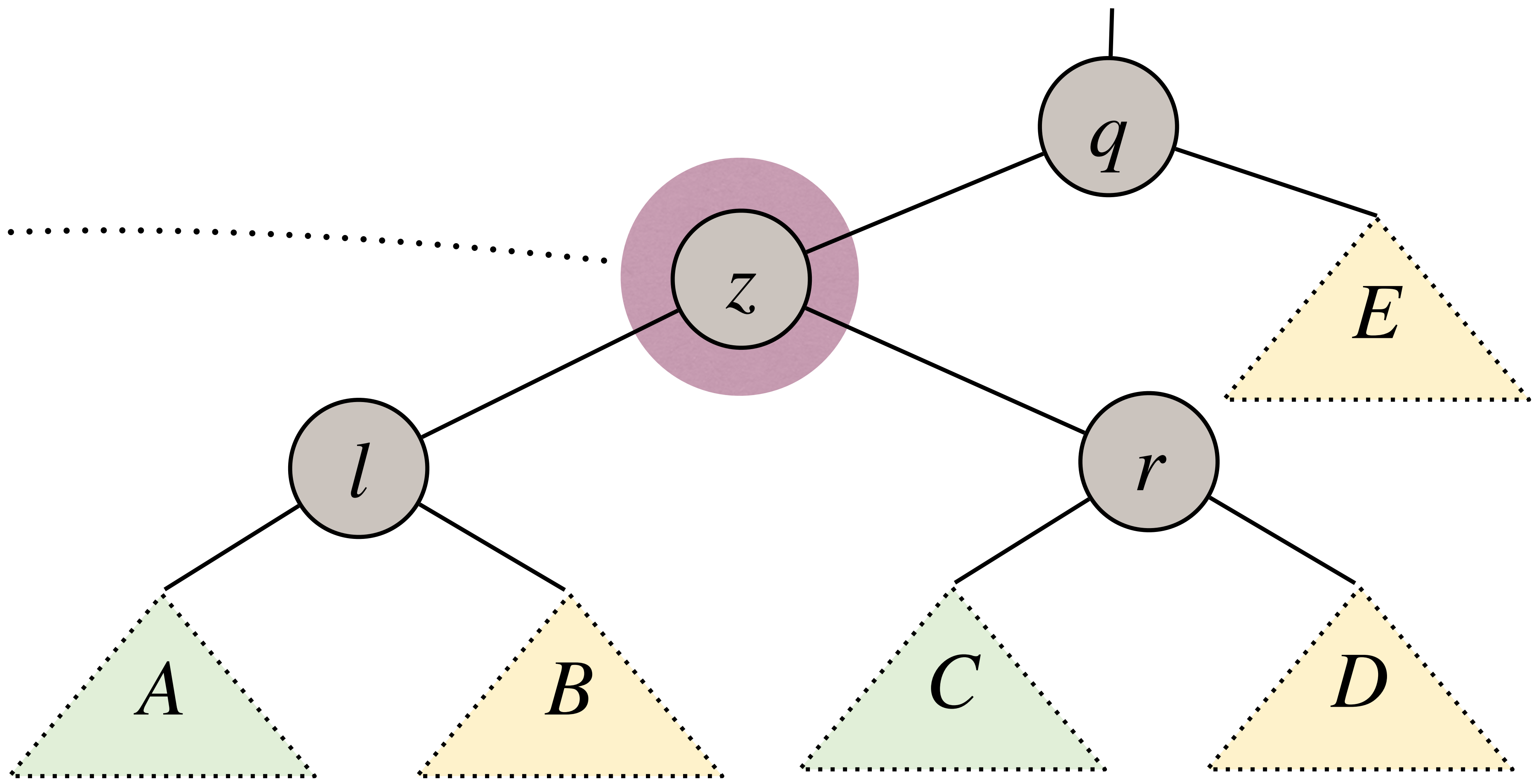


$\ldots, z, A, r, B, q, C, \ldots$ $\ldots, A, r, B, q, C, \ldots$

Remove in BST

- **BSTRemove(T,z):** Remove element z from T. Notice, removal should not break the BST property.
- **Case 3**: z has two children.
  - **Case 3a**: $z.right.left = Null$
  - **Case 3b**: $z.right.left \neq Null$

- Which one should be here to replace node z ?
  - The min value node in subtree rooted at $z.right$.
- That is, replace node z with $BSTSuccessor(z)$.



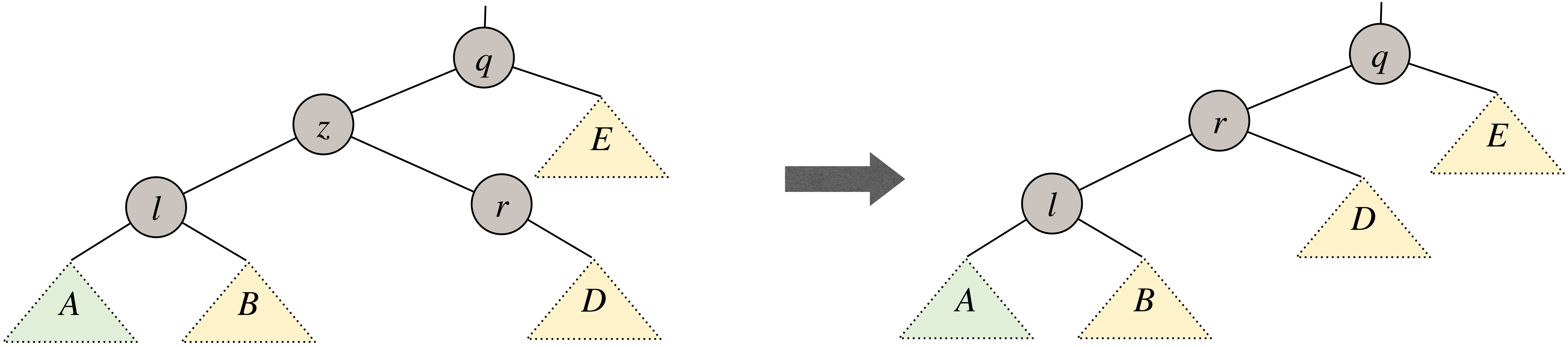
$\ldots, A, l, B, z, C, r, D, q, E \ldots$

- $BSTSuccessor(z)$ can be:
  - r if $r.left = Null$
  - $BSTMin(r.left)$ if $r.left \neq Null$

Remove in BST

- **`BSTRemove(T,z)`**: Remove element z from T. Notice, removal should not break the BST property.

- **Case 3a**: z has two children **and** $z.right.left = Null$



$\ldots, A, l, B, z, r, D, q, E \ldots$

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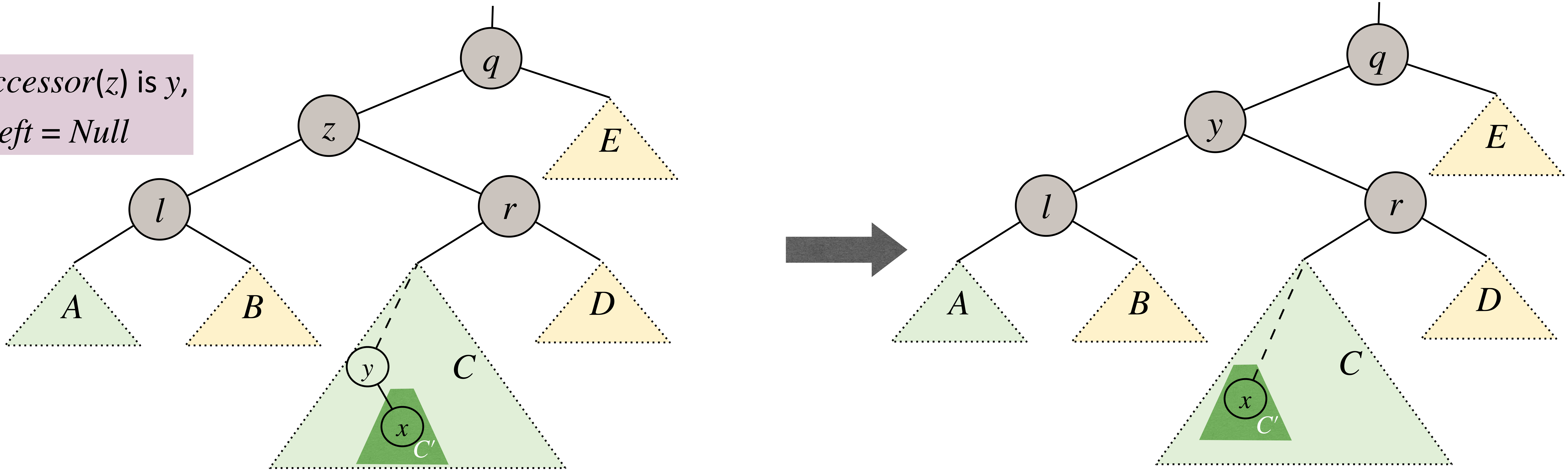
Remove in BST

- `BSTRemove(T,z)`: Remove element z from T. Notice, removal should not break the BST property.

- **Case 3b**: z has two children **and** $z.right.left \neq Null$

$BSTSuccessor(z)$ is y, thus $y.left = Null$



$\ldots, A, l, B, z, y, C', C \backslash C', r, D, q, E \ldots$

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Remove in BST

- `BSTRemove(T,z)`: Remove element z from T. Notice, removal should not break the BST property.
- **Case 1**: z has no child. $\Theta(1)$
  - Easy, simply remove z from the BST tree
- **Case 2**: z has one single child. $\Theta(1)$
  - Elevate subtree rooted at z's single child to take z's position.
- **Case 3a**: z has two children **and** $z.right.left = Null$ $\Theta(1)$
- **Case 3b**: z has two children **and** $z.right.left \neq Null$ $O(h)$

Worst-case time complexity of Remove operation is $\Theta(h)$.

Efficient implementation of Ordered Dictionary

	`Search(S,k)`	`Insert(S,x)`	`Remove(S,x)`
`SimpleArray`	$O(n)$	$O(1)$	$O(n)$
`SimpleLinkedList`	$O(n)$	$O(1)$	$O(1)$
`SortedArray`	$O(\log n)$	$O(n)$	$O(n)$
`SortedLinkedList`	$O(n)$	$O(n)$	$O(1)$
`BinaryHeap`	$O(n)$	$O(\log n)$	$O(\log n)$
`BinarySearchTree`	$O(h)$	$O(h)$	$O(h)$

- BST also supports other operations of **Ordered Dictionary**, in $O(h)$ time.
  - But the height of a n-node BST varies between $\Theta(\log n)$ and $\Theta(n)$.

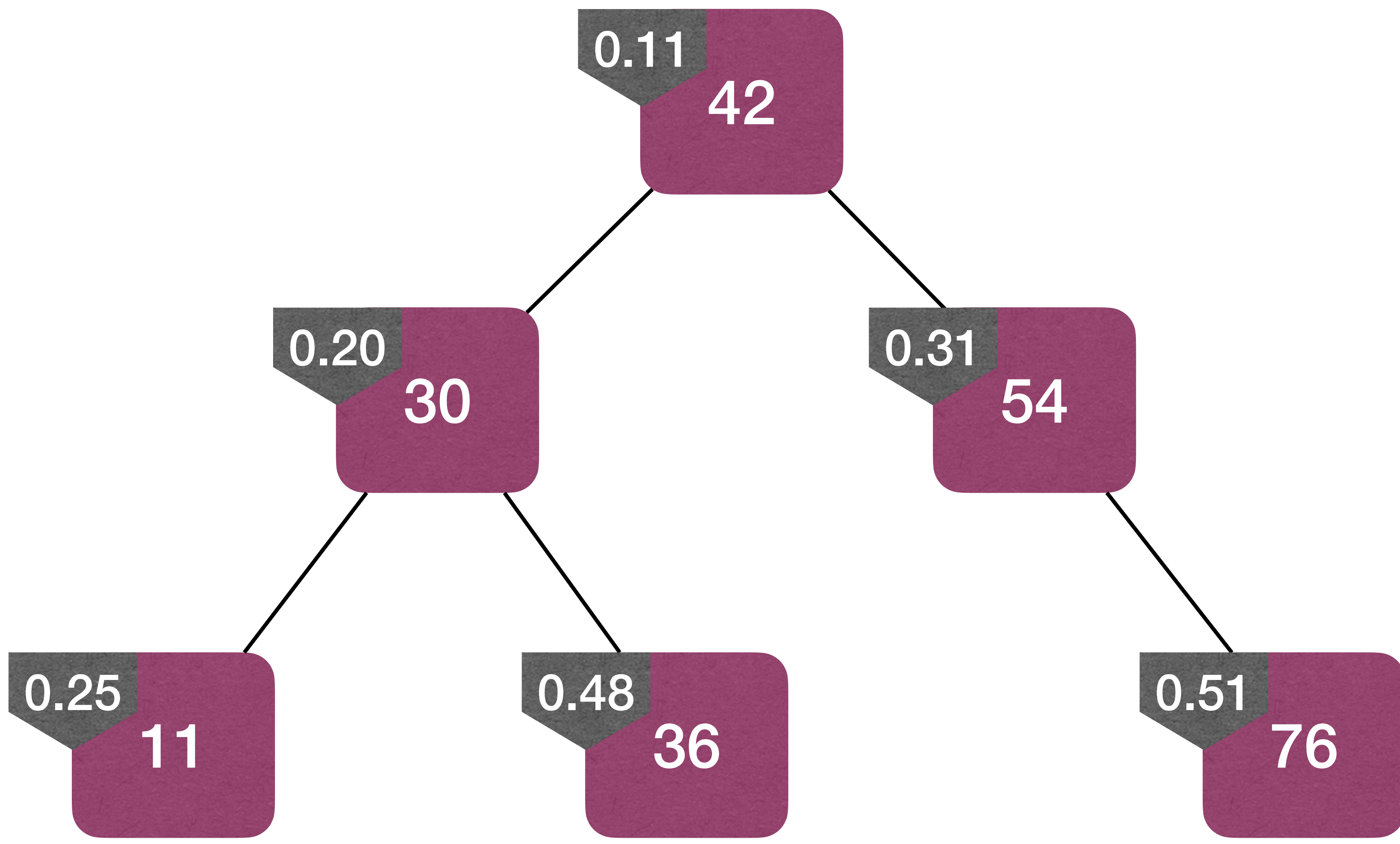
Height of BST

- Consider a sequence of Insert operations given by an adversary, the resulting BST can have height $\Theta(n)$.
  - E.g., insert the elements in increasing order.
- What is the expected height of a randomly built BST? How to build it?
  - Build the BST from an empty BST with n Insert operations.
  - Each of the $n!$ insertion orders is equally likely to happen.
- The expected height of a randomly built BST is $O(\log n)$. Why?

Treaps

Treap: A randomized BST structure

- A **Treap** (Binary-Search-**Tr**ee + He**ap**, 树堆) is a binary tree in which each node has a **key value**, and a **priority value** （usually randomly assigned）.
- The **key values** must satisfy the **BST**-property:
  - For each node y in left sub-tree of x: $y.key \leq x.key$
  - For each node y in right sub-tree of x: $y.key \geq x.key$
- The **priority values** must satisfy the **MinHeap**-property:
  - For each descendent y of x: $y.priority \geq x.priority$



A Treap is not necessarily a complete binary tree. (Thus it is not a **BinaryHeap.**)

Uniqueness of Treap

- **Claim:** Given a set of n nodes with distinct key values and distinct priority values, a **unique** Treap is determined.

- Proof by induction on n:

  - **[Basis]**: The claim clearly holds when $n = 0$.

  - **[Hypothesis]**: The claim holds when $n \leq n' - 1$

Uniqueness of Treap

- **[Inductive Step]**:
  - Given a set of n' nodes, let r be the node with **min priority**. By **MinHeap**-property, r has to be the root of the final Treap.
  - Let L be set of nodes with key values less than $r.key$, and R be set of nodes with key values larger than $r.key$.
  - By **BST**-property, in the final Treap, nodes in L must in left sub-tree of r, and nodes in R must in right sub-tree of r.
  - By induction hypothesis, nodes in L lead to a unique Treap, and nodes in R lead to a unique Treap.

How to build Treap

- How do we build a Treap?
  - Starting from an empty Treap, whenever we are given a node x that needs to be added, we assign a **random priority** for node x, and insert the node into the Treap.
  - Alternative view of an n-node Treap: a BST built with n insertions, in the order of increasing priorities. (**Why?**)
    - Then we only need to worry about BST property if build a Treap in this order.

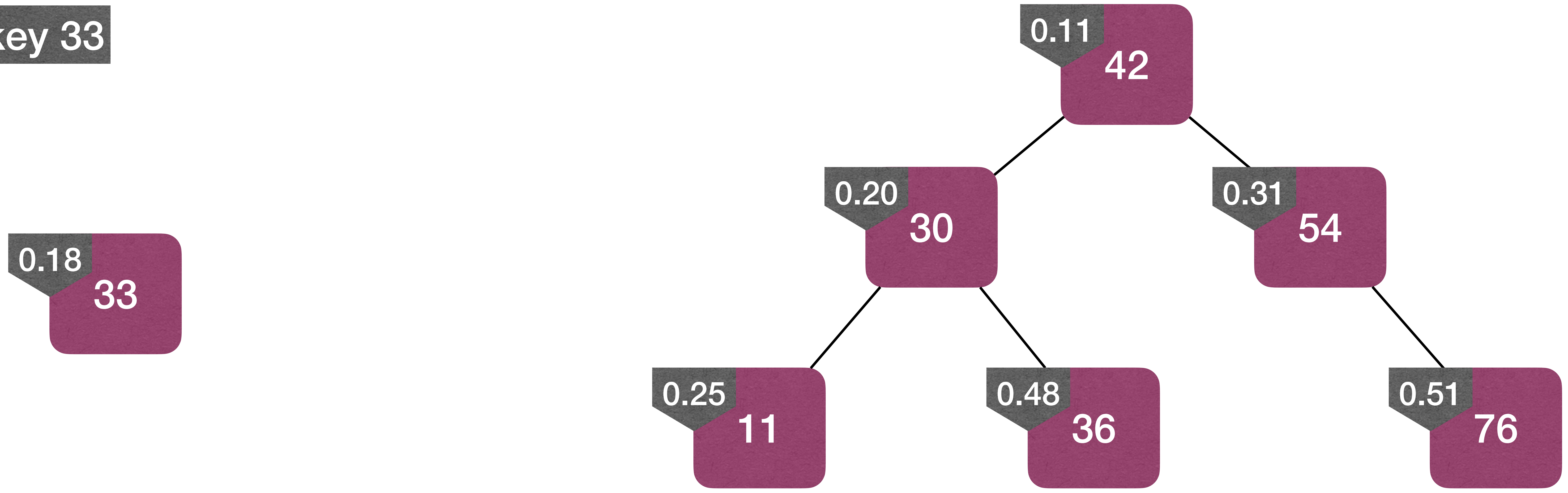
How to build Treap

- **A Treap is like a randomly built BST, regardless of the order of the insert operations!** (Since we use **random priorities!**)
- A Treap has height $O(\log n)$ in expectation.
  - Therefore, all **ordered dictionary** operations are efficient in expectation.
  - Even if the operations are given by an **adversary**!

Insert in Treap

- Step 1: Assign a random priority to the node to be added.
- Step 2: Insert the node following BST-property.
- Step 3: Fix MinHeap-property (without violating BST-property).

Example: Insert element with key 33



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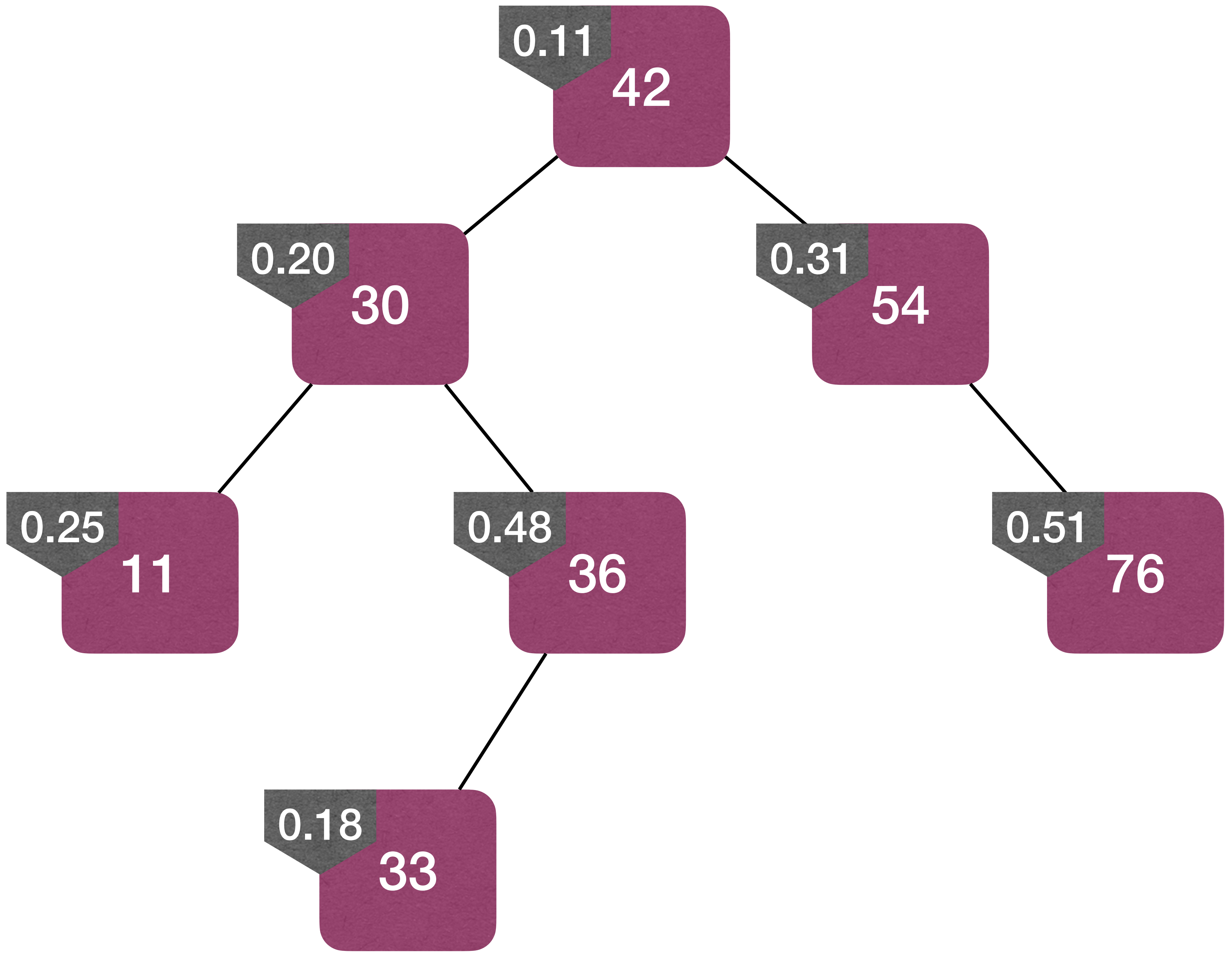

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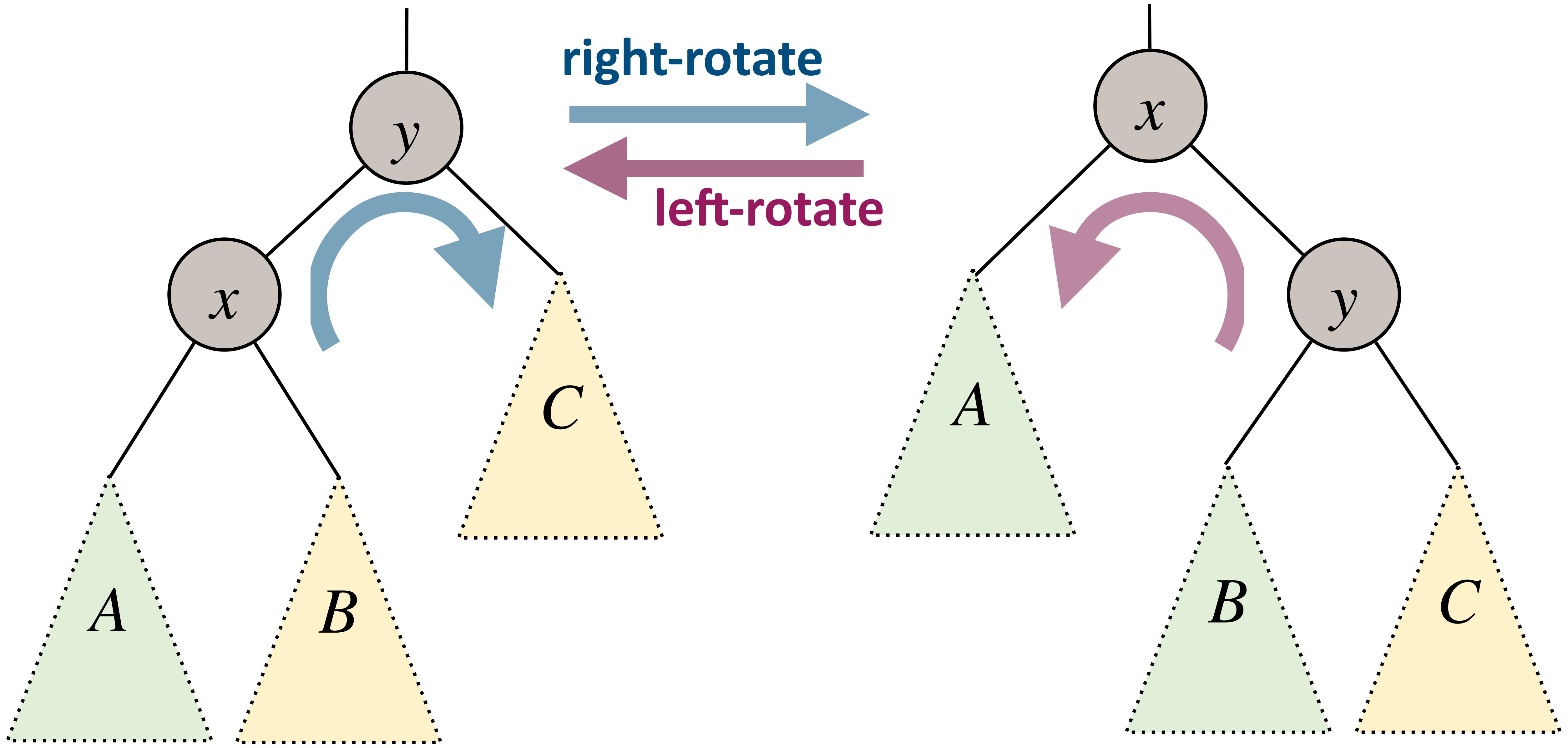
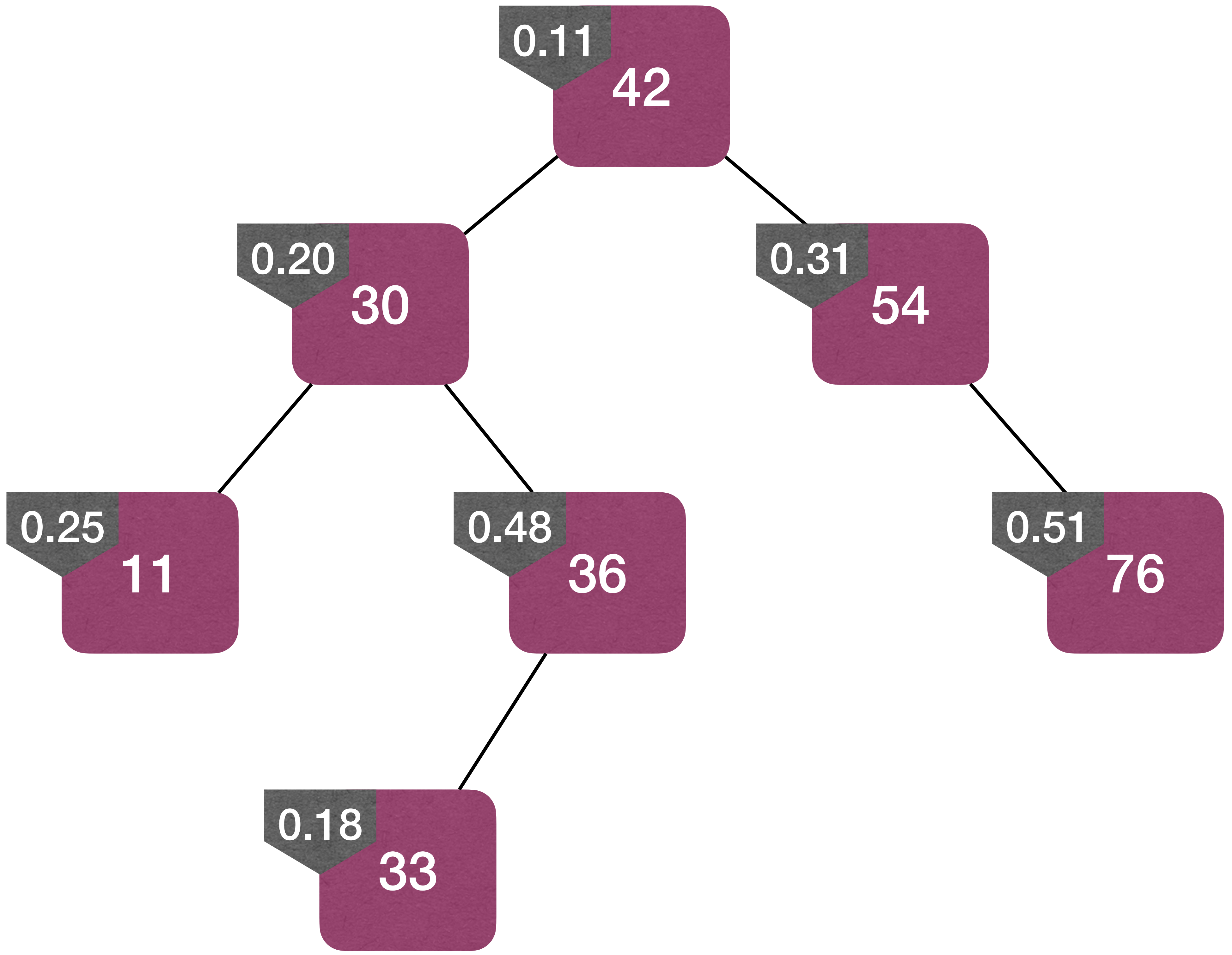


Rotation changes level of x and y, but preserves **BST** property.



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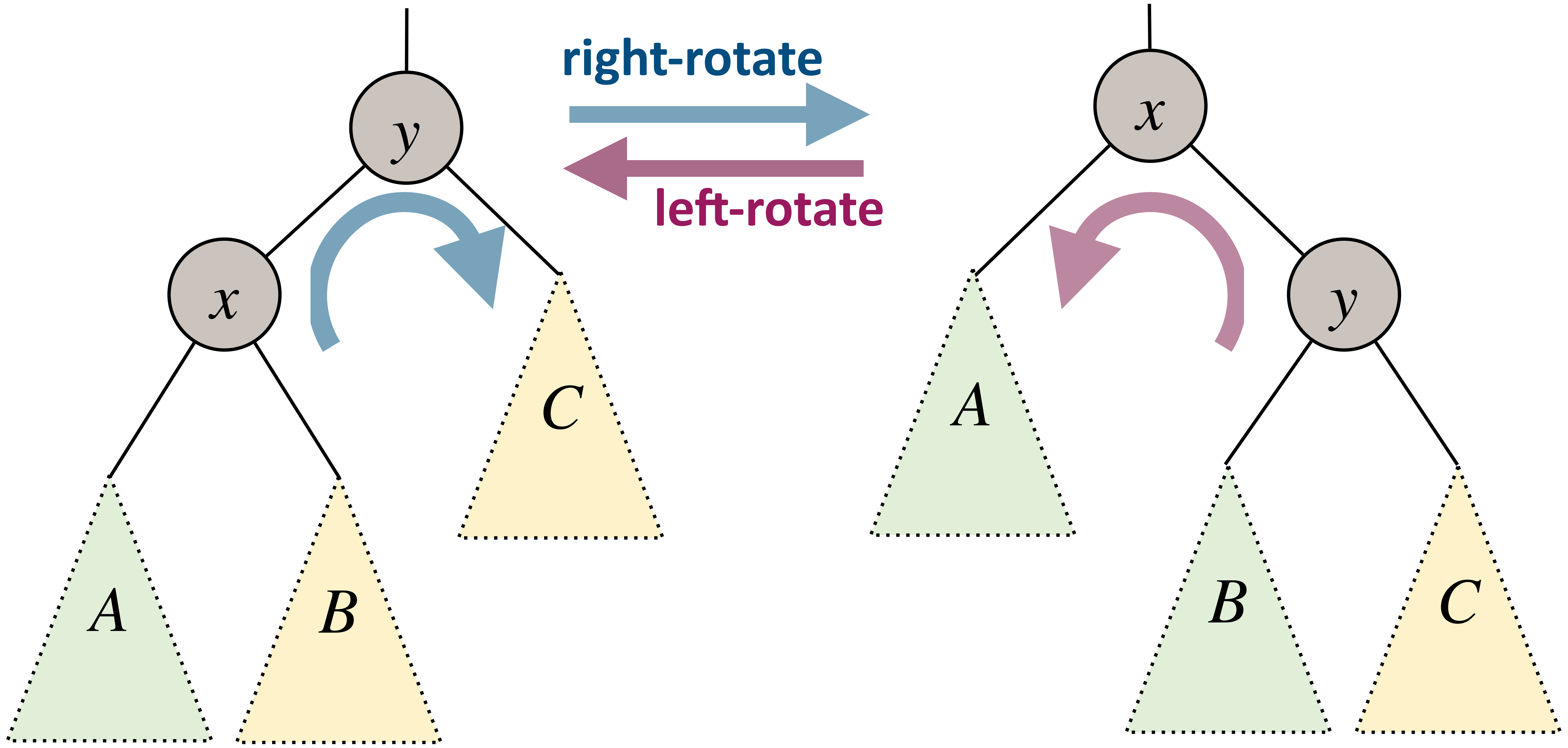

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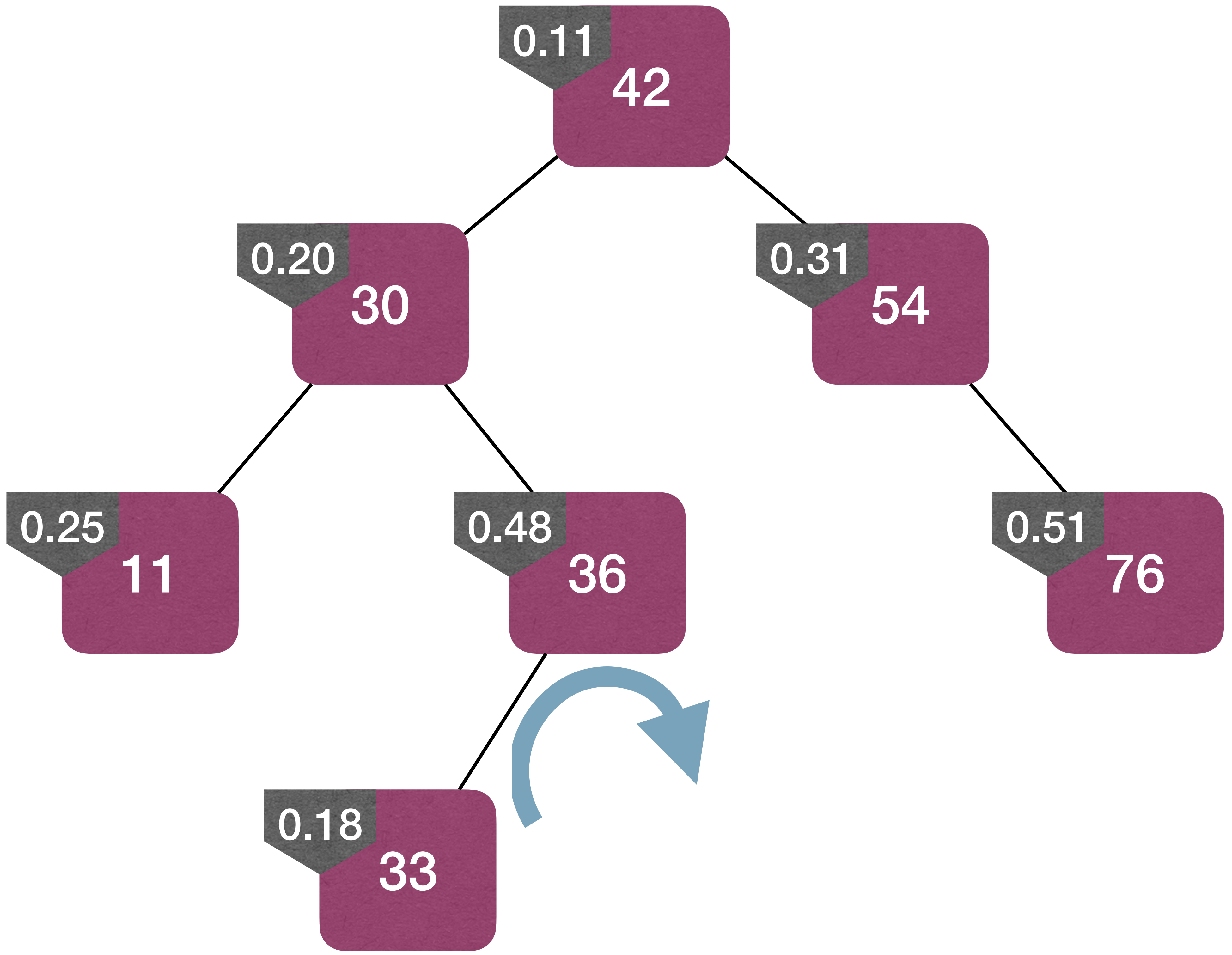

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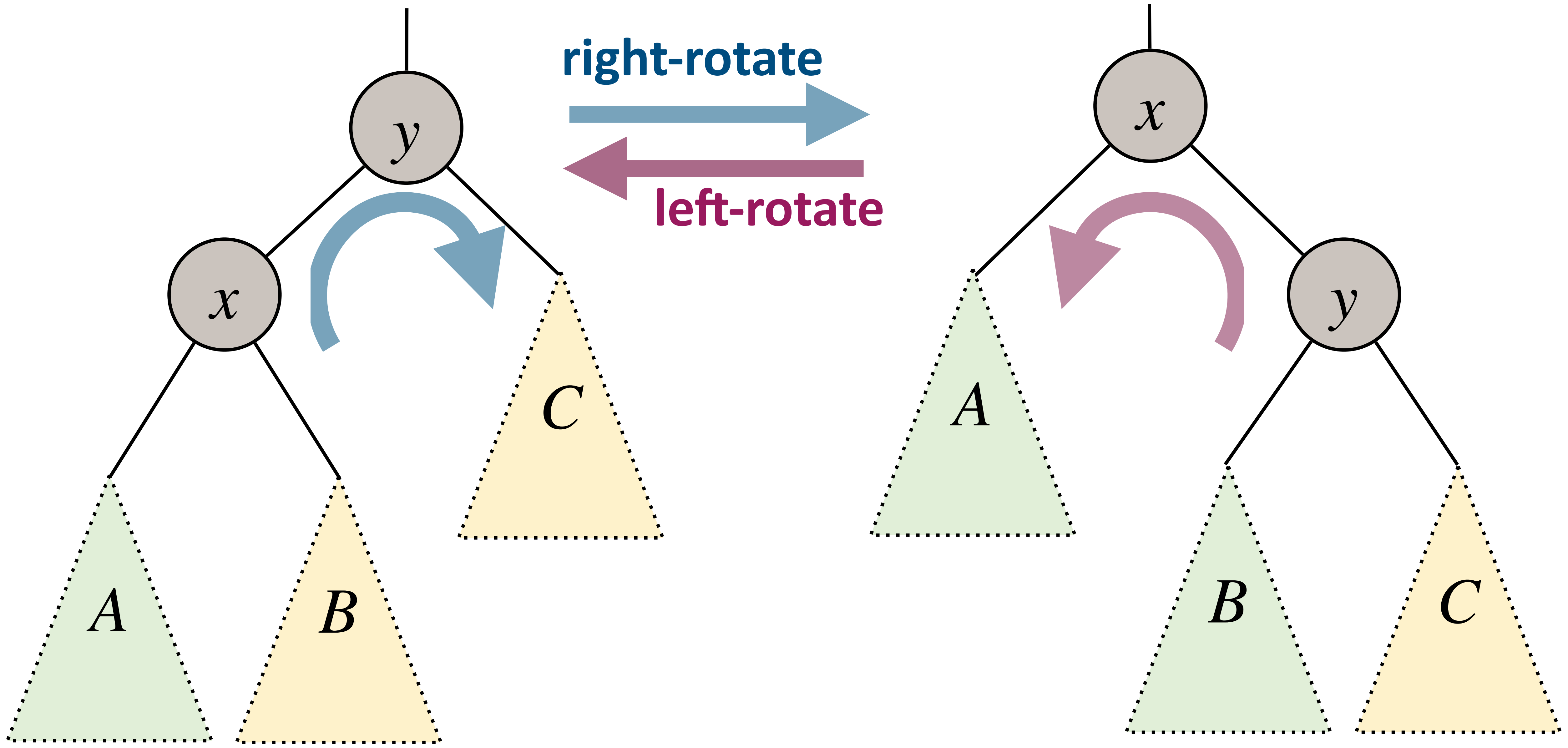

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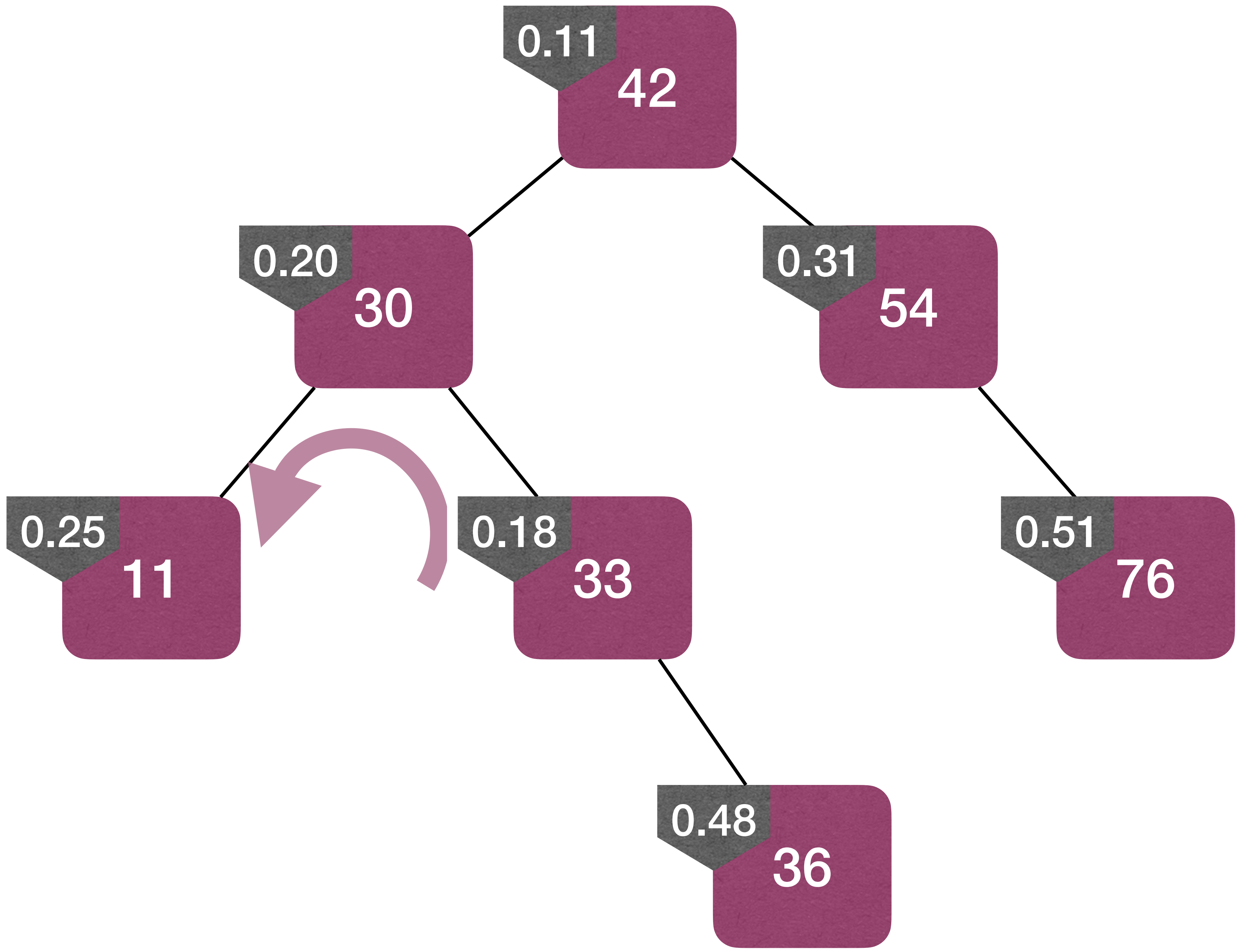

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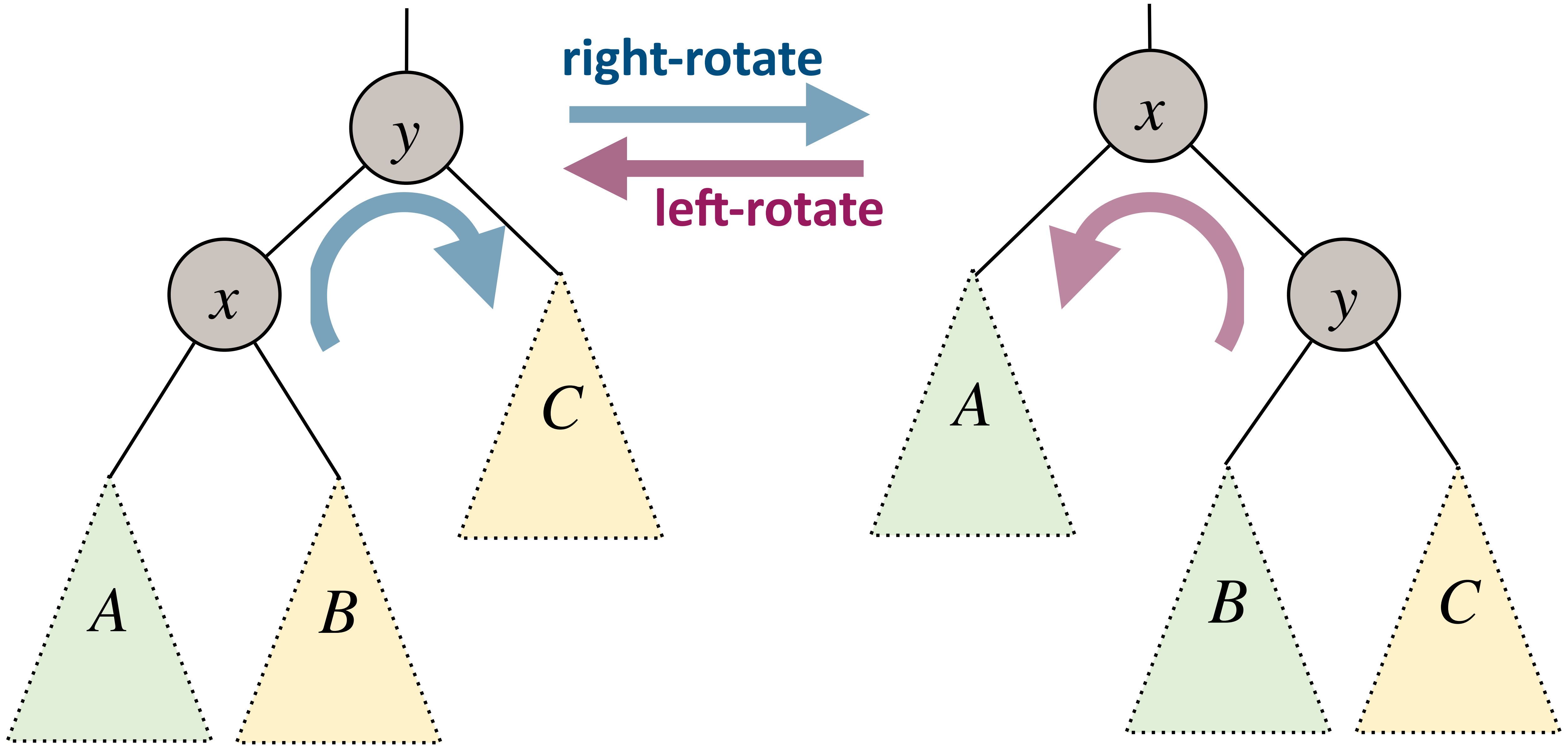

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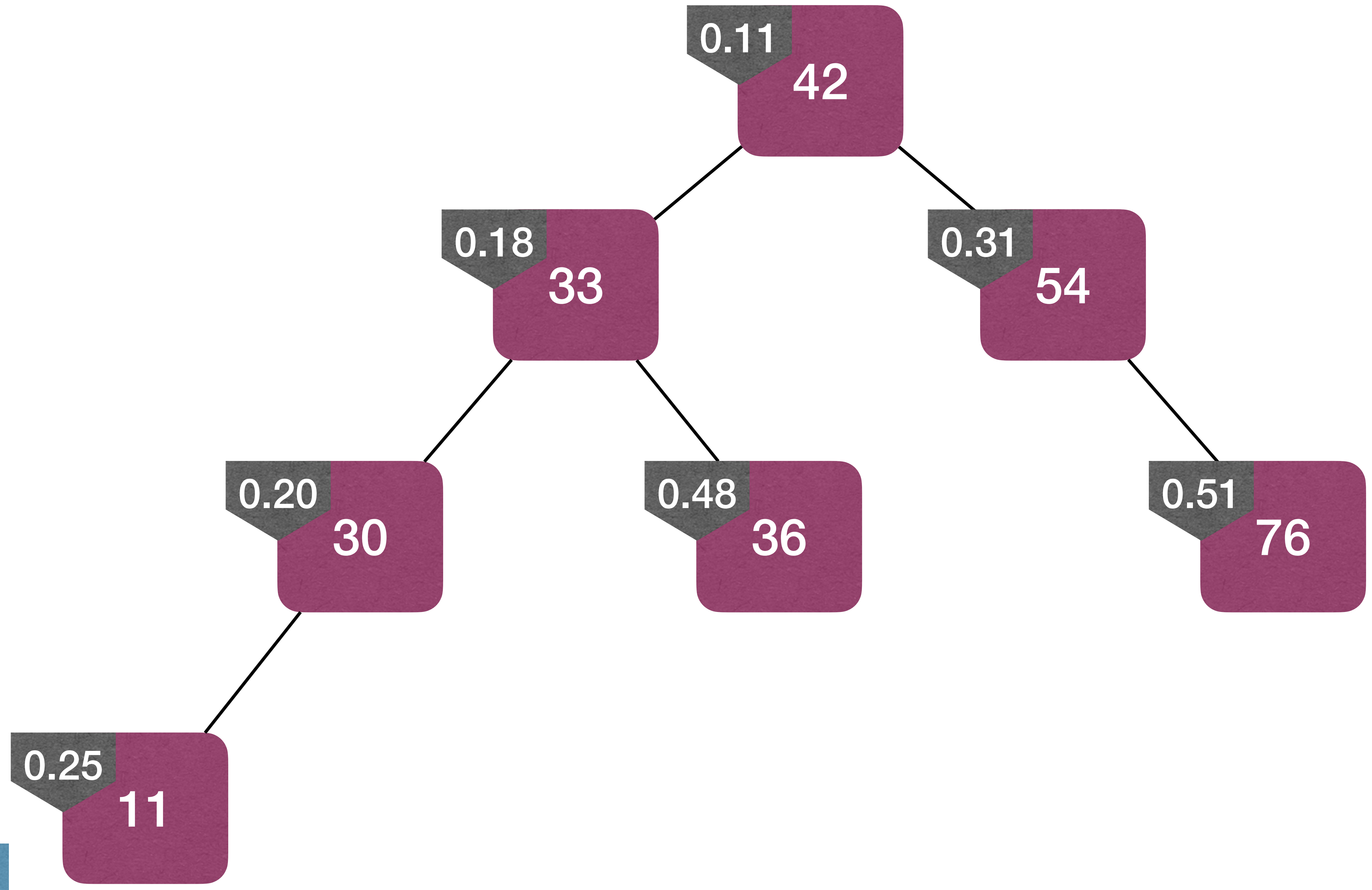

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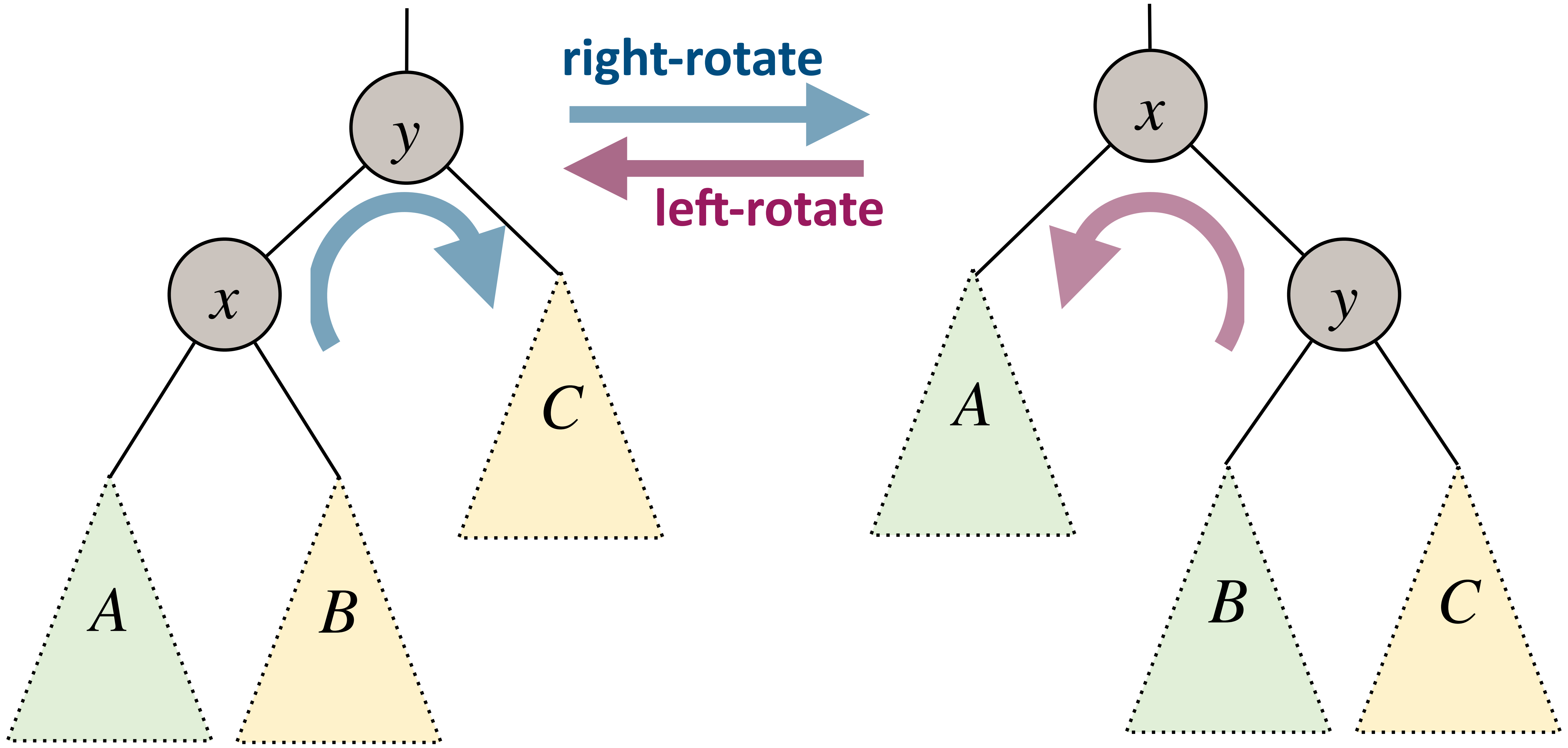
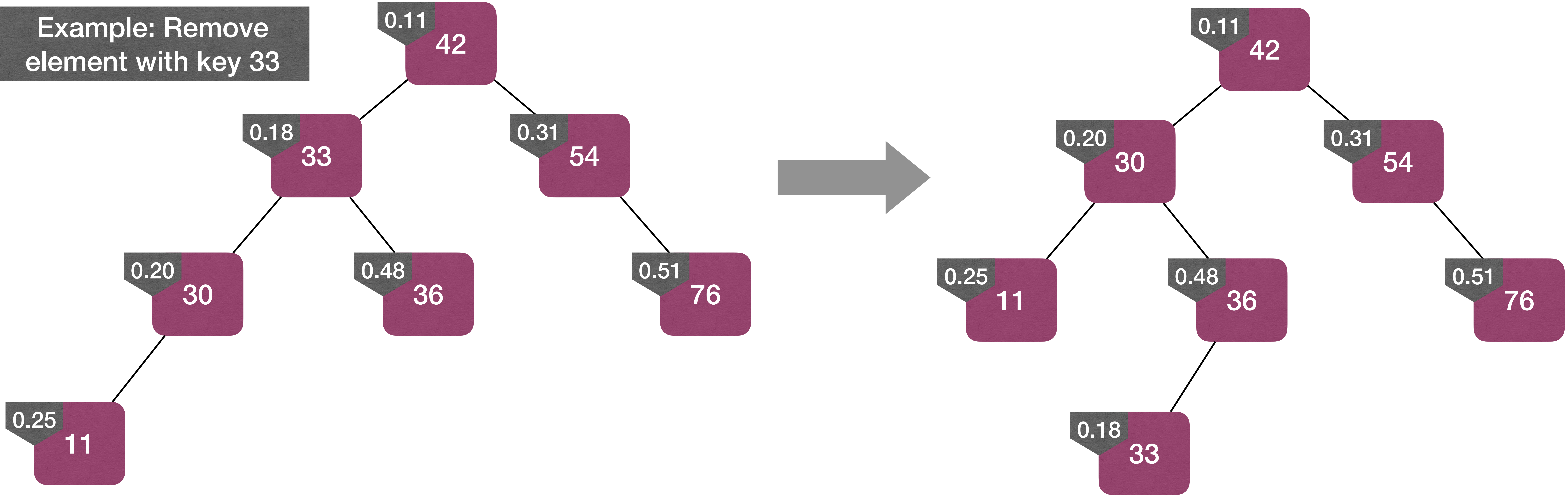


Use rotations to push-up violating nodes until MinHeap-property restored.

Remove in Treap

- Given a pointer to a node, how to remove it? Just invert the process of insertion!
  - Step 1: Use rotations to push-down the node till it is a leaf.
  - Step 2: Remove the leaf.

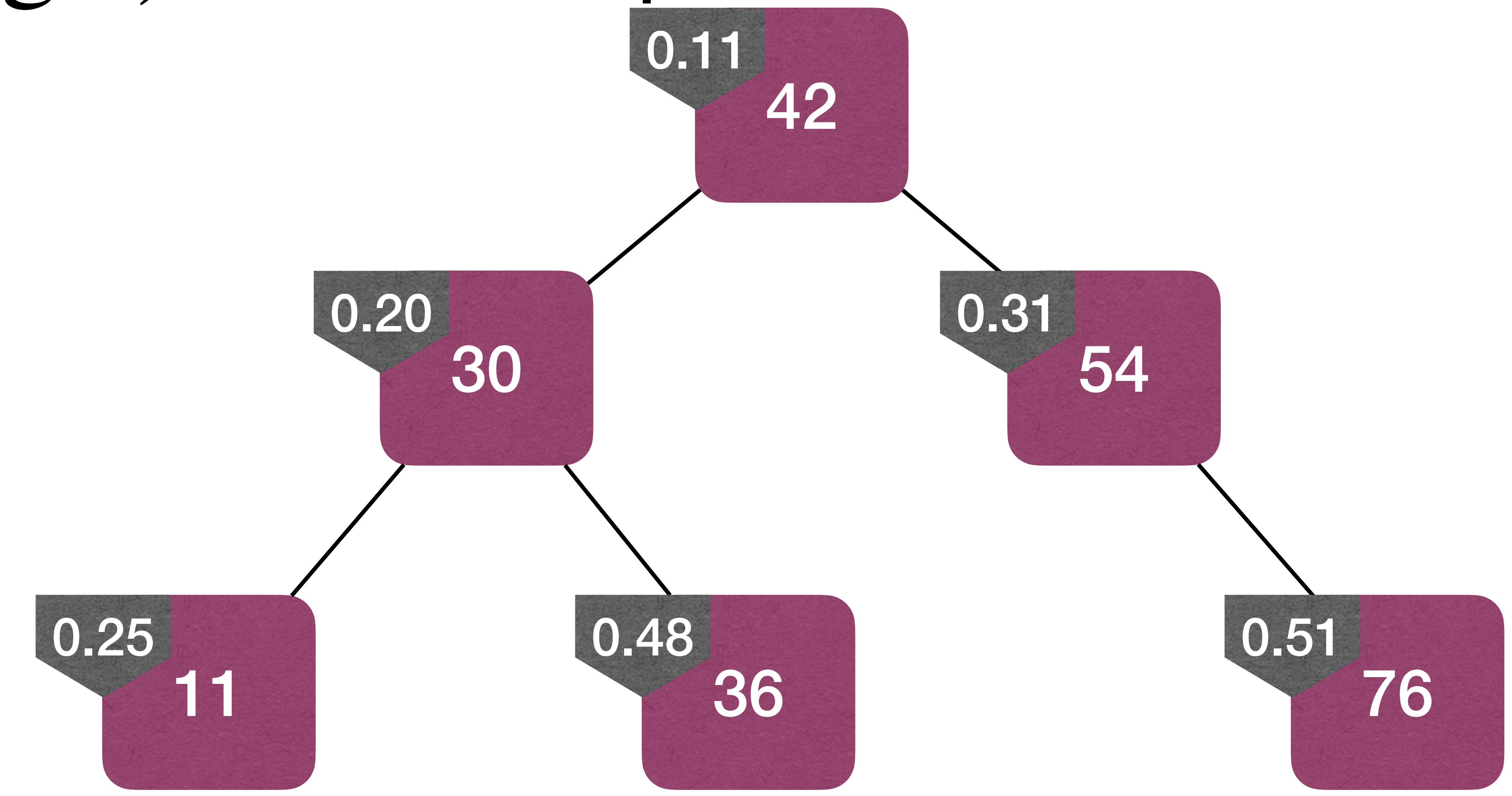
Example: Remove element with key 33



Summary on Treap

- A probabilistic data structure.
- Like a randomly built BST.
  - Expected height is $O(\log n)$ even for adversarial operation sequence.
- Support **ordered dictionary** operations in $O(\log n)$ time, in expectation.

Question: How to design a data structure supporting ordered dictionary operations in $O(\log n)$ time, even in **worst-case**?



Further reading

- [CLRS] Ch.12
- [Morin] Ch.7 (7.2)
- [Sedgewick] Ch.3